Diagnostic and Remedial Teaching in Arithmetic

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PREFACE

This book is written for students of the teaching of arithmetic and for the teacher in the elementary school who is face to face with the question of what can be done to reduce the number of failures in the subject of arithmetic, and to eliminate the difficulties which interfere with pupil progress. This book is not a treatise on the whole subject of the teaching of arithmetic but deals intensively with only one phase of instruction; namely, the techniques for diagnosing pupil difficulties in all phases of arithmetic, and the types of remedial exercises which have been found by experiment to eliminate the difficulties that are located.

The discussion will help teachers to use standardized tests to the greatest advantage. Only those tests that help to illustrate the points under discussion in this book have been described. This was felt to be a more helpful plan than to include descriptions of all available tests. A list of such tests is given in the Appendix.

The uses of standard tests in teaching are discussed in detail from the point of view of their various functions. It is essential that the purposes for which tests are intended be clear to the teacher who wishes to select a test for a particular purpose. Detailed discussions of how tests are given are not included in the discussion since the necessary directions usually are furnished with the tests. It is believed that any teacher who studies carefully the material in the text of this book will have

little difficulty in using any of the tests that have been published.

Special stress is placed on the use of the results of tests as a means of diagnosis, and on the limitations of such procedures. Techniques of detailed psychological diagnosis are described which should be used in all difficult cases to determine the exact nature of the difficulty which is interfering with desirable progress. Test scores indicate the general level of the pupil's ability, and locate the place where a difficulty may be found, but indicate in only a very superficial way the nature of the difficulty. The problem of the teacher is, then, to determine the cause of the difficulty by a careful analysis of the pupil's work and his mental processes, and to prescribe the necessary remedial work.

Much of the illustrative material in the discussion in this book is based on the researches of the author and his graduate students. Some of this has been published in magazine articles, chiefly in the *Elementary School Journal* and in the *Journal of Educational Method*. In some cases illustrative material has been taken from the various manuals which the author has published to accompany his diagnostic tests in the several processes in fractions, whole numbers, and decimals. Significant contributions by other workers in this field have been freely quoted. Certain data from the study by Buswell and John on pupil difficulties in fundamental processes have been rearranged and are given to show the frequency with which the common difficulties occur.

Suggestions for remedial work are included in the various chapters in connection with the discussions of difficulties and the analyses of the elements in each of the processes. It is the author's belief that most of the

difficulties in arithmetic would be eliminated if the right kinds of instructional materials were available. In such materials due consideration would be given to the construction of comprehensive diagnostic tests, to the preparation of instructional materials in which special attention has been given to the step by step development of the learning process, and to the points in each process which have been found by investigation to present serious difficulties to pupils. This applies both to processes in arithmetic, and to procedures in problem solving.

The author acknowledges his indebtedness to many workers in this field who are contributing to the improvement of the teaching of arithmetic. Many of the ideas herein presented are the direct result of the six splendid years of research in Detroit under the direction of S. A. Courtis. Much of the experimental work was done in the Minneapolis Public Schools during the five years that the author was Director of Instructional Research in the schools of that city. The support of principals and teachers was unfailing in the attempts that were made to make the tools that are described in this book effective instruments of instruction. Special acknowledgment is made of the many contributions by the author's graduate students who have dealt intensively with special problems in their individual researches. The author is particularly indebted to Dr. W. E. Peik for a careful reading of the entire manuscript, and to G. O. Banting for helpful suggestions.

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Chapter I

INTRODUCTORY

Surveys of instruction have shown that one of the chief causes of nonpromotion in the elementary school is failure in the subject of arithmetic. Numerous investigations have been made to determine the reasons for this condition. The scientific study of the arithmetic curriculum has presented evidence that much of the traditional subject matter commonly taught in the schools of the past generation is of little social utility, and much too difficult for many children in the grades in which it has been taught. The elimination of cumbersome, difficult, useless processes and topics has greatly reduced the complexity of the skills that the pupils are now expected to acquire. At the same time the work in arithmetic has been greatly enriched and vitalized by stressing its application in meaningful activities and situations. Other investigations have revealed the inadequacy of much of the instructional material found in textbooks. To overcome this deficiency the newer textbooks are being constructed according to more or less adequate specifications based on extensive studies of pupil difficulties in the subject and careful analyses of the steps in the learning process. The simplification of the arithmetic curriculum and the improvement of instructional materials will undoubtedly do much to reduce the extent of failure in arithmetic.

1. THE RESULTS OF EARLY ARITHMETIC INVESTIGATIONS.

By means of tests of achievement much less reliable than the well standardized tests that are now available, Rice[1] and Stone[2] early revealed the wide differences in the achievements of pupils in different communities and of pupils in the same grade. A large amount of overlapping of scores was found from grade to grade, some studies revealing a range of ability of as much as six or seven years in a single grade. In the earlier studies of test results, special stress was placed on the differences in the achievements of classes, and relatively little attention was given to the causes of the wide variations in ability of individual pupils.

In the course of time, careful consideration of the implications of the data secured by means of the tests resulted in the formulation of various plans of adapting instruction to differences in the abilities of pupils. In some of the progressive schools, pupils were grouped according to their level of progress to reduce the range of ability in classes. However, the variation in performance after a period of teaching was almost as great as it had been before the grouping. Parallel track plans were also devised which made it possible for pupils to progress at different rates. Burk devised a plan for completely individualizing instruction in arithmetic and other subjects, so that each pupil could progress at his own rate of speed. This plan is the basis for the Winnetka[3] plan of

[1] J. M. Rice, "Educational Research: A Test in Arithmetic," *Forum*, Vol. 34, pp. 437–52.

[2] C. W. Stone, "Arithmetic Abilities and Some Factors Conditioning Them," *Teachers College Contributions to Education*, No. 19 (New York: Columbia University, 1908).

[3] C. W. Washburne, *Twenty-Sixth Yearbook of the National Society for the Study of Education*, Part I (Bloomington, Illinois: Public School Publishing Company, 1926), pp. 219–29.

individualized instruction. None of these plans has solved the problem of providing for individual differences in instruction in arithmetic, since in almost every class there are still found pupils who fail to make progress at a desirable rate.

2. EMPHASIS ON THE ANALYTICAL STUDY OF ARITHMETIC PROCESSES.

One of the early studies of Stone[1] showed that arithmetic is made up of a large number of specific abilities each of which must be developed through careful practice. For example, Stone[1] showed that a pupil might be doing fairly satisfactory work in addition but be quite deficient in several of the other operations. The result of this and similar studies by Brueckner,[2] Thorndike,[3] Knight,[4] and others has been the detailed analysis of the specific abilities and skills which constitute the complexity of processes in arithmetic. The earlier tests of achievement have been greatly improved by making them more analytical. Instead of lumping many skills in a single test, carefully constructed diagnostic tests in each process have been devised by means of which the teacher can locate the specific weakness in a process which may be causing difficulty for a pupil. Instructional materials are being greatly improved because of the increasing body of information available regarding the elements in a process that must be presented during the learning period. Special consideration is also being given to the clarifica-

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[1] Stone, *loc. cit.*

[2] L. J. Brueckner, *Manual for Diagnostic Test in Fractions* (Minneapolis, Minnesota: Educational Test Bureau, 1926).

[3] E. L. Thorndike, *Psychology of Arithmetic* (New York: The Macmillan Company, 1922), pp. 51–101.

[4] F. B. Knight, E. M. Luse, and G. M. Ruch, *Problems in the Teaching of Arithmetic* (Iowa City, Iowa: Iowa Supply Store, 1925).

tion of some of the steps in the several processes which investigations have found to be especially difficult for pupils. This is being done by means of carefully constructed practice exercises in which the difficult elements are isolated and presented one after the other in such a way that the difficulty is almost eliminated. The necessity for repeated review in order that the specific skills that have been taught may be retained is being recognized in the preparation of various types of practice exercises which embody inventory tests, diagnostic tests, and remedial exercises.[1]

In recent years, important investigations have been concerned with the study of the nature of the psychological difficulties of pupils who are not making satisfactory progress in arithmetic. These studies give definite information as to the most common types of errors made by pupils, their habits of work, the faulty procedures they use, the nature of the difficulties they encounter in problem solving, their general physical and mental characteristics, and their social qualities.

Techniques have been devised as a result of these studies for locating the specific source and nature of these deficiencies, and for analyzing the mental processes of pupils in working examples which are causing difficulty. The information thus secured has been used to improve instructional materials by the preparation of special types of exercises on elements in processes which are known to present unusual difficulty to pupils; by providing exercises which teach economical, efficient procedures from the beginning; by providing carefully constructed diagnostic tests to locate specific difficulties. Adequate

[1] L. J. Brueckner, C. J. Anderson, G. O. Banting, and E. Merton, *Diagnostic Tests and Practice Exercises in Arithmetic* (Philadelphia: The John C. Winston Company, 1929), Grades 3, 4, 5, 6, 7, and 8.

remedial exercises to be used to provide the practice necessary to raise the work to a satisfactory standard have also been devised.

3. CONSIDERATION OF INDIVIDUAL DIFFERENCES.

The fact of individual differences in the needs of pupils and in their rates of learning is being taken into consideration in the teaching procedures used in the classroom. To this end inventory tests have been devised for use at the beginning of the year by means of which the pupils and teachers can determine the general needs of each individual. Practice exercises have been prepared which enable the teacher to assign the remedial work needed by each pupil. Many of these standard practice exercises are so constructed that pupils can progress at different rates, each pupil according to his ability. Standardized survey tests are being used to measure the relative achievements of classes from time to time. Diagnostic tests of a very specific kind are being used to determine the exact nature of difficulties of a class or of individuals. In cases in which the nature of the deficiency cannot be determined by a test or by an analysis of the pupils' written work, more refined diagnostic methods are used to discover whether the difficulty may not be due to faulty associations, incorrect, involved procedures, roundabout methods of work, or some other reason that can be discovered only by methods approximating those used in individual psychological exminations.

It is the purpose of this book to contribute to a constructive attack on the teaching of arithmetic by bringing together in a compact form: 1, the results of the important studies that have been made to develop adequate, reliable methods of determining the achievement of pupils

in arithmetic; 2, scientific techniques for diagnosing the nature of the difficulties and deficiencies that may prevent satisfactory progress in the various processes and in problem solving; and 3, the types of remedial and practice exercises that may be used to remove specific types of difficulties, and to raise the work in arithmetic to a satisfactory level.

In the discussion of these points, the important types of achievement tests now available are presented and their uses described. Detailed analyses of the basic skills in each of the processes in whole numbers, fractions, and decimals are presented for the purpose of showing the care that must be taken in the construction of instructional materials to insure the inclusion of all of the elements involved in each of the above-mentioned processes. These analyses of basic skills are shown to be the basis of diagnostic tests which should be used to determine the specific nature of the pupil's deficiencies in each process. Detailed reports of the common faults revealed by a diagnostic study of the work of pupils deficient in the several processes and in problem solving are given, and the techniques that can be used to locate these faults are discussed. A knowledge of the sources and causes of greatest difficulty will help the teacher to give the pupils special help at critical points.

4. The Basis of a Well-Rounded Program in Arithmetic Instruction.

The following list of items suggests the basis of a well-rounded program of arithmetic instruction:

(1) Knowledge of specific objectives and functions of arithmetic.

(*a*) Using materials and processes having social value.

(*b*) Showing how number has aided in the systematizing of the quantitative aspects of the environment.
(*c*) Developing an appreciation of the informational function of arithmetic.
(*d*) Developing an appreciation of number as a means of precise and accurate thought.
(*e*) Having an appreciation that arithmetic is in fact a social study of great importance.
(*f*) Assisting pupils to understand quantitative references in readings.
(*g*) Developing adequate concepts of units of measurement.
(*h*) Training pupils in the use of reference materials.
(*i*) Selecting social applications of value.

(2) Recognition of the function of general method in the teaching of arithmetic.
(*a*) Knowing how to organize subject matter into large units for study by the class.
(*b*) Securing a learning situation in which motives of a relatively high order are present.
(*c*) Providing conditions in which the growth of desirable social characteristics, attitudes, and ideals will take place.
(*d*) Recognizing the possibilities of socializing experiences in the lesson.
(*e*) Enriching and vitalizing the experiences of pupils.
(*f*) Recognizing arithmetic applications in other subjects, such as history, geography, science, and health work.
(*g*) Using experiences and activities that arise in connection with local conditions and situations.
(*h*) Providing intercorrelations between subjects in situations in which number functions.

(3) Procedures in presenting new processes.
(*a*) Knowing the steps in the learning process.
(*b*) Showing the social utility of the new step in a process through applications and problems.

(c) Using previously acquired skills in presenting a new step.
(d) Presenting only one new step at a time.
(e) Considering special difficulties in the new step.
(f) Providing adequate practice on the new step.
(g) Giving special help to those pupils for whom the class presentation is not adequate.
(h) Teaching efficient methods of work.
(i) Providing for the repeated review of the new step in subsequent lessons to insure retention of the new skill.

(4) Procedures in diagnosing pupil difficulties.
(a) Locating the specific needs of the pupils in the class.
(b) Knowing the most common pupil faults and weaknesses in processes.
(c) Using suitable diagnostic procedures.
(d) Preparing informal diagnostic exercises.
(e) Teaching pupils to diagnose their own difficulties.
(f) Adjusting the class work in such a way that the teacher has time available for diagnosis of pupil difficulties.
(g) Selecting pupils in need of intensive diagnostic study.
(h) Keeping records of diagnosis.
(i) Filing the results of standard diagnostic tests.
(j) Using graphs, etc., to interpret shortcomings.
(k) Using school records to locate other causes of deficiencies.
(l) Considering the pupil's personality as a factor in diagnosis.
(m) Interpreting the results of the diagnostic study.

(5) Procedures in providing remedial instruction.
(a) Selecting the remedial exercises adapted to the needs of each individual.
(b) Organizing group work as the basis for instruction.
(c) Eliminating faulty habits, such as counting, etc.
(d) Correcting faulty, inefficient methods of work.

(*e*) Teaching methods of checking all work.
(*f*) Using pupils whose work is satisfactory to assist deficient pupils.
(*g*) Devising methods of showing pupils their improvement.
(*h*) Teaching pupils how to select the proper remedial exercises.
(*i*) Providing for variations in rates of pupil progress.
(*j*) Evaluating the effectiveness of the remedial work by tests.
(*k*) Setting up reasonable standards of attainment.
(*l*) Teaching pupils good techniques of problem solving.

(6) Methods of making the work vital and meaningful for the pupils.
(*a*) Teaching pupils to purpose to improve.
(*b*) Using materials of the proper level of difficulty.
(*c*) Motivating the drill work.
(*d*) Making pupils conscious of objectives.
(*e*) Helping pupils to diagnose their difficulties.
(*f*) Helping pupils to overcome their difficulties.
(*g*) Using graphs, charts, etc., to show achievements.
(*h*) Using games.
(*i*) Providing activities in which arithmetic processes are applied concretely.
(*j*) Showing the life need of processes prior to teaching them.
(*k*) Using illustrations from daily life to show application of processes.
(*l*) Encouraging original applications of processes.
(*m*) Showing applications of processes in other subjects.
(*n*) Using excursions, exhibits, and projects.
(*o*) Showing social significance of quantitative concepts.
(*p*) Providing opportunities for exploring topics of special interest.
(*q*) Keeping of notebooks on topics being studied.
(*r*) Assigning topics for special reports.
(*s*) Distributing opportunity for participation in class activities.

(t) Organizing the contents of the subject matter to be taken up into coherent units stressing applications and functions.

(7) Methods of socializing the work in the class period.

(a) Providing for group or committee work on special topics.
(b) Providing opportunities for pupils to assist in organizing and planning class activities and discussions.
(c) Using superior pupils to assist those below standard.
(d) Assigning special topics for research.
(e) Encouraging practical applications of processes in the home and elsewhere.
(f) Organizing the aspects of the school bank, milk supply, etc., in which arithmetic is used in such a way that pupils participate in phases of the activity.
(g) Considering questions of thrift in the home, in school, etc.
(h) Using a socialized form of recitation.
(i) Considering the social significance of such topics as money, the number system, etc.
(j) Arousing interest and developing habits of extensive reading on topics being considered by the class.
(k) Developing an appreciation of the social significance of quantitative relations.
(l) Posting papers of special merit.
(m) Encouraging courtesy on the part of pupils whose work is completed before that of the others.
(n) Requiring neatness in all written work.

(8) Use of tests in instructing.

(a) Standard tests.
1. Giving of tests.
2. Scoring of tests.
3. Tabulating test results.
4. Interpreting test results.
5. Considering limitations of standard tests.
6. Charting test results.
7. Using test results as the basis for group, for remedial work, etc.

8. Using the results of standard diagnostic tests.
9. Utilizing provisions that enable the teacher to secure test materials.
10. Filing of findings and test papers.
11. Selecting tests suited to the purpose of the teacher.

(*b*) Informal tests.

1. Constructing informal tests—giving informal tests.
2. Scoring informal tests.
3. Using informal tests effectively, as inventory, pre-test, re-test, etc.
4. Tabulating informal test results.
5. Interpreting test scores.
6. Considering the limitations of test scores.
7. Varying the form of test used.
8. Using test scores as the basis for remedial work, diagnosis, etc.
9. Filing informal tests for future use.
10. Using school equipment, such as school mimeograph, etc., in preparing tests.
11. Preventing pupils from cheating in tests.
12. Coöperating with other teachers in the preparation of tests.

(*c*) Use of pertinent data from other tests.

1. Considering intelligence test scores in diagnosis.
2. Using reading test scores as basis for diagnosis of difficulties in problem solving.

(9) Points of a general character.

(*a*) Filing of materials of instruction such as drill cards, etc.

(*b*) Keeping of records of pupil progress.

(*c*) Planning of work, preparation of materials to be used, etc.

(*d*) Preparing improved types of supplementary materials.

(*e*) Distribution of time to problem and formal drill work.

(*f*) Keeping available materials essential in an ideal modern program of arithmetic.
(*g*) Using the course of study for reference and guidance.
(*h*) Using the textbook as a teaching tool.
(*i*) Reading important scientific contributions on aspects of arithmetic.
(*j*) Handling details of method technique.
(*k*) Supervising pupil's work during supervised study periods.

5. THE STEPS THAT CHARACTERIZE WELL-CONCEIVED REMEDIAL WORK.

The steps which characterize well-conceived work with remedial cases in the classrooms are concisely stated in the *Twenty-Fourth Yearbook of the National Society for the Study of Education*, Part 1, as follows:

1. Discovery of deficiency in the course of classroom activities.

2. More intensive observation and study of the exact nature of the difficulties encountered in regular class work.

3. Individual examination by means of personal interview and selected standardized and informal tests with a view of revealing fundamental attitudes and causes of deficiency.

4. Formulation of specific remedial exercises which attack the causes of deficiency.

5. Initiation of regular remedial work in a manner to enlist pupil coöperation and effort.

6. Measurement with records, notes on pupil reactions, and study of progress.

7. Adjustment of work to changing needs until the deficiency is removed.

CHAPTER II

THE USES OF TESTS IN MEASUREMENT AND DIAGNOSIS

Arithmetic is a subject that lends itself readily to analytical treatment as far as the elements involved in computation in arithmetic processes are concerned. Arithmetic is made up of a hierarchy of habits, specific skills, and general abilities. Each may be isolated, studied independently, and have its elements determined by critical analysis. This fact has long been recognized by those who have attempted to evaluate the results of instruction by means of educational tests and to adapt instruction to the needs and capacities of pupils as revealed by these tests.

At the present time there are available several different kinds of standardized tests in arithmetic. Standard survey tests make it possible to secure information as to the general status of pupils in achievement in arithmetic as a whole, or in some single phase of the subject, such as addition of fractions or subtraction of decimals. Diagnostic tests have been prepared to locate the particular element in a process or to determine the type of example which causes difficulty. Some of these diagnostic tests are supplemented by carefully prepared blanks, several of which are included on the following pages, to be used for recording the specific difficulties and faults found by an individual clinical examination of pupils who are markedly deficient in arithmetic. To facilitate diag-

nosis, these blanks contain lists of the most common arithmetic difficulties which extensive studies of the work of deficient pupils show to exist. In this chapter the place of tests in teaching arithmetic processes is discussed in detail.

1. Elements in Arithmetic Ability.

Five important elements, similar to those proposed by Thorndike[1] for intellect in general, must be considered in measuring the ability of pupils, as a basis for diagnostic and remedial work:

1. Their *rate* of work.
2. Their *accuracy* of work.
3. The *altitude* or level of difficulty of the most difficult examples that they are able to work correctly.
4. The *area* of the skills they have mastered in a process.
5. Their *methods of work.*

(a) **Rate of work.** It is a recognized fact that one of the important characteristics of a skill is the speed with which an individual can perform the tasks involving it. Other things being equal, the greater the speed the greater is the skill possessed. Speed of response is not stressed in the initial stages of the learning process, since in the beginning it is important that the steps in the operation itself should be learned and a correct procedure established. Speed of response should increase with mastery of the operation. It is also a function of maturity. Rate of work in arithmetic is easily determined by finding the number of examples of a given type that a

[1] E. L. Thorndike, *Measurement of Intelligence* (New York: Bureau of Publications, Columbia University, 1928).

pupil can work in a certain period of time. Usually tests given for the purpose of determining rate of work should contain more examples than the most rapid worker in a class can complete in the time allowed. However, various factors affect the pupils' performance on these tests, such as their speed of writing, the amount of effort put forth, the understanding of the directions, degree of interest, their state of health, and similar factors.

Courtis and Thorndike[1] have shown that marked changes in scores result from corrections made for differences in rate of writing on tests in which there is a large amount of writing, especially in the primary grades where pupils have inadequate control over the writing process. A pupil who can write rapidly may make a higher score on such a test than a pupil who cannot write so rapidly but who in fact has greater arithmetical ability. While therefore the real ability of the pupils cannot always be determined, their relative ability can be reliably inferred, except in a few cases, from their performance in a standardized rate test.

Rate of work is a factor that has determined the time standards on practice exercises, such as those of Courtis and Studebaker, in which pupils are required to practice on a certain unit of work until they can complete it in a given length of time. The number of minutes is reduced from grade to grade because of the need of allowing for the factor of maturity and its influence on speed of work. The basic assumption underlying this application of time standards is that pupils in a particular grade should not be required to increase their speed of work beyond the reasonable limits which pupils of their maturity can nor-

[1] S. A. Courtis and E. L. Thorndike, "Correction Formulae for Addition Tests," *Teachers College Record*, Vol. 21, pp. 1–24.

mally reach. It has often been necessary for teachers to modify the time standards in the cases of individual pupils who for various reasons, such as slowness of reaction and general inability to speed up their work, are unable to complete an exercise in the time limits that have been set up, although their computations are accurate but very slow and no undesirable habits of work can be discovered. On more difficult exercises, such as column addition and long division, the practice of allowing enough time on a test to find the answer and also to check the work is sometimes followed. Too much stress has in many cases been placed on speed of work with the result that accuracy has been made a minor objective. There is no reason why time standards cannot also take into account the time needed for checking the work.

(b) Accuracy of work. A knowledge of the pupil's rate of work is of little value unless this information is accompanied by a record of his accuracy of work. A pupil may work a large or a small number of examples with a relatively high or low degree of accuracy.

There are various methods of measuring the pupil's accuracy of work. One method is to find the per cent of the examples attempted that are worked correctly. A pupil who worked correctly nine of twelve examples attempted would receive a rating of 75 per cent accuracy. When standard norms are available, the relative status of a pupil can be determined by comparing his accuracy rating with these norms.

Another method of determining a pupil's ability is to find the number of examples that he has worked correctly regardless of the number of examples that he has attempted. Thus two pupils who both work eight exam-

ples on a test correctly receive the same score, although one attempted only eight examples and worked all correctly while the other attempted to work fifteen of the examples and worked only eight of them correctly. If the examples in the test are all equally difficult it is obvious that the two pupils have quite different habits of work, one being slow but accurate, the other being a fast but inaccurate worker.

Clearly the ability of the pupil cannot be found by considering either his rate of work or his accuracy, one independent of the other. Both factors must be considered together. The following chart contains the possible combinations in which relative degrees of rate and accuracy may be found in classes and in individual pupils:

Superior rate of work and a high degree of accuracy	Average rate of work and a high degree of accuracy	Low rate of work and a high degree of accuracy
Superior rate of work and an average degree of accuracy	Average rate of work and an average degree of accuracy	Low rate of work and an average degree of accuracy
Superior rate of work and a low degree of accuracy	Average rate of work and a low degree of accuracy	Low rate of work and a low degree of accuracy

(c) **The level of difficulty.** It is desirable to know not only the rate and the accuracy with which a pupil can work examples of a certain type, but also the level of his progress in the subject. A pupil's level of ability in

arithmetic can be found by discovering the highest level of difficulty of the examples in the various processes which he is able to work correctly. This is found in much the same way that we find the mental capacity of a pupil by such a series of graded tasks as are found in such tests as the Stanford Revision of the Binet-Simon Test or the Kuhlmann Individual Mental Test. A series of items is arranged in a scale in the order of their difficulty from very easy to very difficult. (See page 28.) The pupil is given approximately as much time as he needs to solve all of the examples he is able to work. The level of the pupil's ability is roughly determined by the difficulty of the examples of greatest difficulty he is able to work correctly. The pupil who is able to work examples high on the scale has a higher level of ability than a pupil who cannot work them. Rate of work is not considered in finding the level of the pupil's general ability in arithmetic. However, the ability of the pupil at a particular level can be found by giving him tests made up of examples similar to the elements at that level and finding his rate of work and accuracy.

SCALE OF PROBLEMS BASED ON THORNDIKE'S C-A-V-D[1] ARRANGED IN ORDER OF DIFFICULTY

1. A nickel is 5 cents. How many nickels make 15 cents?
2. How much must you add to 7 to make 10?
3. How many cents make half a dollar?
4. Dick is 11 years old. John is 15. How much older is John than Dick?
5. George is 9 years old. How old will he be in 2 years?
6. How many cigars can you buy for $1 at the rate of three for a quarter?
7. John had 3 dollar bills, 7 dimes and 2 pennies. How much is that?

[1] Completion; Accuracy; Vocabulary; Directions.

8. At $12\frac{1}{2}$ cents each, how much more will 6 tablets cost than 10 pencils at 5 cents each?

9. What number added to 16 gives a number 4 less than 27?

10. A man bought land for $400. He sold it for $445, gaining $15 an acre. How many acres were there?

11. What part of 16 equals half of 24?

12. I bought $4\frac{1}{2}$ yards of cloth, gave the clerk $2, and received 20 cents as correct change. What was the price of the cloth per yard?

13. 8 is $1\frac{1}{3} \times$ ——.

14. A man spent two thirds of his money and had $8 left. How much had he at first?

15. How many times as long as 8 feet is 12 yards?

16. Write in the space left for it the number that should come next.

240, 120, 60, 30, ——

17. If a snail crawls at the rate of an inch in $1\frac{1}{6}$ minutes, how long will it take it to go 8 feet?

18. By how much must you increase 10 so that it will stand in the same ratio to 8 that 15 does to 10?

19. A ship has enough provisions to last her crew of 500 men 6 months. How long would they last 1200 men?

20. Three men hired a pasture for $24; the first put in 2 horses, the second put in 3 horses, and the third put in 4 horses; how much ought each to pay?

In measuring the level of the pupil's ability in such a process as addition of fractions, the teacher must, therefore, consider the difficulty of the examples included in the test. Examples ranging from very easy to fairly difficult should be included in test exercises given for the purpose of providing a basis of grouping pupils according to their ability.

The principle of difficulty should assist in the grading of the content of the curriculum and in the preparation of special exercises on points of extreme difficulty. The

elimination, at least for the slower pupils, of certain of the usual requirements, which are very difficult and are not commonly used in life, should also result from the application of this principle.

In order to secure an adequate picture of the arithmetic ability of the class, the teacher must secure measurements of achievement at different levels of difficulty by means of tests which consider both rate and accuracy of work at the several levels. The results of a test consisting of relatively easy examples of a low level of difficulty will not give adequate, valid information as to what the same pupils will achieve in a test of subject matter of a considerably higher level of difficulty.

(d) **The area of the skills in a process.** The ability of a third-grade pupil in such a process as addition of whole numbers can be adequately determined only by means of a test containing, as far as is possible, all of the various types of examples in which that process is used. The same principle applies to all of the other processes in whole numbers, fractions, decimals, per cent, and denominate numbers. This is true especially during the period when the several operations are first being taught. This is the critical time in the learning process.

The present practice of measuring the ability of the pupil by means of a few examples selected at random from the complete process is not satisfactory. In such tests, the results do not show what the pupil's performance will be on the types of examples not contained in the test. Such tests usually do not take into account the information now available as to the complexity of skills and abilities which constitute a particular process. The

ability of the pupils to work all of the various types of examples which make up the *complete area* of the process must be inferred from his score on a test of only a small part of the process. This is a highly unreliable way of locating possible deficiencies in specific types of examples. Detailed discussions of the applications of the concept of *area* are given in the following chapters dealing with the different operations. Obviously, at the time a particular process is being learned, special consideration must be given to the variety of types of examples in that process. Especially in the preparation of the practice exercises, it is essential to make certain that the pupil is brought into contact with the many combinations in which the various elements and skills are found which constitute the complete process.

(e) **The pupil's methods of work.** At any level of difficulty and for any process an important element to consider in evaluating the pupil's work is his methods of work. The pupil may make satisfactory scores on tests but at the same time be working considerably below his possible maximum level of achievement because of faulty habits of work. These faulty habits may be vocalizing his procedure, counting by various methods, lack of neatness in writing out his work, incorrect statement of procedure, and similar faults which tend greatly to reduce his efficiency. These faults can be discovered through an observation of his methods of work. Efficient methods of work can be substituted for these faults which tend to invalidate any test results which purport to give a measure of the pupil's arithmetic ability. Emotional instability, nervousness under test conditions, the tendency to work as if under a strain, dawdling, inattention,

and similar psychological conditions can also be discovered through observation.

2. Types of Tests and Their Functions.

(a) **Survey tests in arithmetic.** Survey tests in arithmetic are used to determine how the achievements of a class or of individuals in the class compare with standards or norms of accomplishment. These survey tests are of three types: 1, rate tests; 2, scales or power tests; 3, curriculum tests.

(1) Rate tests. The purpose of rate tests is to determine how many examples of one particular type, or of varied types, the pupils can work in a given time and to compare their achievement, as measured by both rate and accuracy or by either of these factors, with standard scores. There are several tests of this type that are widely used, one of the best known series being the Courtis Standard Research Tests in Arithmetic, Series B. Typical examples from each part of the test are given on page 23. These tests consist of sets of examples in each of the four fundamental operations with integers. In each test the examples are of the same type, and are of equal difficulty. The pupil's score is determined by the number of examples he attempts and by the per cent he can work correctly in a given time. Tentative June class standards are given by Courtis as follows:

Grade	Addition		Subtraction		Multiplication		Division	
	Tried	Per Cent Correct	Tried	Per Cent Correct	Tried	Per Cent Correct	Tried	Per Cent Correct
4	7.4	64	7.4	80	6.2	67	4.6	57
5	8.6	70	9.0	83	7.5	75	6.1	77
6	9.8	73	10.3	85	9.1	78	8.2	87
7	10.9	75	11.6	86	10.2	80	9.6	90
8	11.6	76	12.9	87	11.5	81	10.7	91

ILLUSTRATIVE EXAMPLES FROM THE COURTIS STANDARD RESEARCH TESTS—SERIES B

Eight of the Twenty-four Addition Examples

You will be given eight minutes to find the answers to as many of these addition examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

127	996	237	386	186	474	877	587
375	320	949	463	775	787	845	685
953	778	486	827	684	591	981	452
333	886	987	240	260	106	693	904
325	913	354	616	372	869	184	511
911	164	600	261	846	451	772	988
554	897	744	755	595	336	749	559
167	972	195	833	254	820	256	127
554	119	234	959	137	533	258	323

Eight of the Twenty-four Subtraction Examples

146246252	80630266	124485018	107419373
52160891	68164329	73098624	65345405
37953635	137825921	152695030	178976226
23913884	62729490	85612816	98060303

Ten of the Twenty-five Multiplication Examples

3268	4795	7954	2386	9745
95	83	74	88	59
6283	9624	7853	4926	5873
47	503	35	620	49

Eight of the Twenty-four Division Examples

$24\overline{)6984}$	$95\overline{)85880}$	$36\overline{)10440}$	$87\overline{)81867}$
$78\overline{)62868}$	$42\overline{)17682}$	$63\overline{)26460}$	$59\overline{)50799}$

The basis of the standards now used by Courtis[1] is described in the following quotation:

> The speeds (rates) set as standard are approximately the average speeds (rates) at which the children of the different grades have been found to work when tested at the end of the year. For any one grade, a random selection of five thousand scores from children in schools of all types and kinds is used as a basis of judgment.
>
> Standard accuracy is perfect work, one hundred per cent. This is a tentative standard only, as there is available very little information in regard to the factors that determine accuracy and the effects of more efficient training.
>
> At present in addition and multiplication, it is only very exceptional work in which the median rises above eighty per cent accuracy, while in subtraction and division the limiting level is ninety per cent.
>
> Standard speeds (rates) are not likely to change greatly. Standard accuracy is surely destined to approach much more nearly one hundred per cent than present work would indicate.
>
> Standard scores are not only goals to be reached; they are limits not to be exceeded. It seems as foolish to overtrain a child as it is to undertrain him. All direct drill work should, in the judgment of the writer, be discontinued once the individual has reached standard levels. If his abilities develop further through incidental training, well and good; but the superintendent who, by repeated raising of standards, forces teachers and pupils to spend each year a larger percentage of time and effort upon the mere mechanical skills, makes as serious a mistake as the superintendent who is too lax in his standards.

By comparing the scores of a pupil, or of a class, with these standards it is possible to determine the general level of achievement and the processes on which addi-

[1] S. A. Courtis, Third, Fourth, and Fifth Annual Accountings (Detroit, Michigan: Department of Coöperative Research, 1913-16).

tional practice is needed. For example, a fourth-grade class might have a median score on the addition test of 8.4 examples tried and 61 per cent correct; in rate of work this class would be somewhat above the standard for rate but below the standard for accuracy.

The following suggestions will assist in interpreting pupil and class scores on this test or on similar rate tests:

Type	Scores		Interpretation Probable Meaning
	Rate	Accuracy	
1	High	High	Marked ability.
2	Average	Average	Normal ability, usual class work sufficient to maintain present level of work.
3	Low	Low	Lack of native ability, or marked defects in training; a carefully organized remedial program is needed here.
4	High	Low	Shows poor training or poor ability; must be given special training in accuracy and taught to check work.
5	Low	High	Shows excellent training; group may be of low native ability, or accuracy may have been stressed at expense of speed; probably could improve speed with special practice.

Rate tests in arithmetic have also been prepared which make it possible to measure the achievements of pupils in a single test on a variety of examples in one or more processes. A typical series of tests of this type suitable for inventorying the work at the beginning of the fourth grade is given on page 26.

Inventory Tests in the Fundamentals[1]

These tests will help you to find how well you remember work like that which you had in the third grade.

[1] Triangle Arithmetics, Book 1, Part 2, page 3. The John C. Winston Company, Philadelphia.

Addition (5 minutes, not including copying)

	a	b	c	d	e	f
1.	14 8	7 46	3 4 4	7 9 6	75 61 32	70 40 90
2.	36 48 53	876 908 309	965 178	798 459 547	243 928 385	$69.89 84.96 73.69

Subtraction (8 minutes, not including copying)

1.	945 824	906 405	547 347	823 114	873 487	720 467
2.	900 246	7,020 2,854	5,914 5,907	5,475 4,957	13,225 4,664	$300.00 48.32

Multiplication (7 minutes, not including copying)

1.	18 3	70 6	621 4	804 2	706 8	45 9
2.	930 6	800 7	641 5	724 3	978 9	$5.49 8

Division (10 minutes, not including copying)

1.	3)36	4)76	2)75	5)255	6)516	9)874
2.	5)1,100	6)865	8)1,960	7)749	9)1,832	8)$16.48

Practice on the process in which your work was weakest.

Each of these tests contains examples of different types in a single process. The time limits that are given suggest the length of time in which pupils should be able to work and check the examples in each test. Inability of the pupil to work all of the examples in a process correctly within these time limits shows that practice is needed in that process. It should be noted that these tests do not contain all types of examples in each process,

thereby limiting their value as a means of exact diagnosis. They must therefore be supplemented by other analytical tests of a more detailed kind containing a wider variety of types of examples. This is essential in order that the teacher may determine the ability of the pupil to work types other than those included in the test. Such tests are described in detail on the pages following.

The basic principle of the rate test, namely, the time element, has been incorporated in such standard practice exercises as the Courtis Standard Practice Tests in Arithmetic, The Studebaker Economy Practice Exercises, and Brueckner, Anderson, Banting and Merton Diagnostic Tests and Practice Exercises for Grades 3, 4, 5, 6, 7, and 8. In these exercises pupils are required to practice on a given unit until they can work all of the examples correctly in a specified time. The length of time decreases from grade to grade.

Other well-known survey tests are the Cleveland-Survey Arithmetic Tests, Compass Survey Tests, and Monroe General Survey Scales in Arithmetic.

(2) Scales. Scales differ from rate tests in two respects: 1, the time limits for the tests are so liberal that each pupil is given practically as much time as he needs to complete all of the examples on the test. His rating is not determined by the *speed* with which he can work the examples but by the number of examples on the scale that he can work correctly; and 2, the examples in the scale are arranged in the order of difficulty from very easy types, on which few pupils fail, to those which only a few pupils can work correctly. The difficulty has been determined by the proportion of pupils who can work the examples correctly. An example missed by only a few pupils in a group is placed lower in the scale than an

example more frequently worked incorrectly by the pupils in the same group. In theory a score on a scale indicates the level of difficulty at which a pupil is just able to work an example correctly. A typical problem scale is given on page 18.

THE WOODY ARITHMETIC SCALES

PUBLISHED BY

TEACHERS COLLEGE, COLUMBIA UNIVERSITY

ADDITION SCALE A

City..............County..................School................Date..............

Name.....................................When is your next birthday?.....

How old will you be?..............Are you a boy or a girl?............

In what grade are you?..............Teacher's name..........................

(1)	(2)	(3)	(4)	(5)	(6)	(7)
2	2	17	53	72	60	3 + 1 =
3	4	2	45	26	37	
	3					

(8)	(9)	(10)	(11)	(12)	(13)
2 + 5 + 1 =	20	21	32	43	23
	10	33	59	1	25
	2	35	17	2	16
	30			13	
	25				

(14)	(15)	(16)	(17)	(18)	(19)
25 + 42 =	100	9	199	2563	$ .75
	33	24	194	1387	1.25
	45	12	295	4954	.49
	201	15	156	2065	
	46	19			

(20)	(21)	(22)	(23)	(24)
$12.50	$8.00	547	$\frac{1}{3} + \frac{1}{3} =$	4.0125
16.75	5.75	197		1.5907
15.75	2.33	685		4.10
	4.16	678		8.673
	.94	456		
	6.32	393		
		525		
		240		
		152		

(25)
$\frac{3}{8} + \frac{5}{8} + \frac{7}{8} + \frac{1}{8} =$

(26)
$$\begin{array}{r} 12\frac{1}{2} \\ 62\frac{1}{2} \\ 12\frac{1}{2} \\ \underline{87\frac{1}{2}} \end{array}$$

(27)
$\frac{1}{3} + \frac{1}{4} + \frac{1}{2} =$

(28)
$\frac{3}{4} + \frac{1}{4} =$

(29)
$$\begin{array}{r} 4\frac{3}{4} \\ 2\frac{1}{4} \\ \underline{5\frac{1}{4}} \end{array}$$

(30)
$$\begin{array}{r} 2\frac{1}{2} \\ 6\frac{3}{8} \\ \underline{3\frac{3}{4}} \end{array}$$

(31)
$$\begin{array}{r} 113.46 \\ 49.6097 \\ 19.9 \\ 9.87 \\ .0086 \\ 18.253 \\ \underline{6.04} \end{array}$$

(32)
$\frac{3}{4} + \frac{1}{2} + \frac{1}{4} =$

(33)
$$\begin{array}{r} .49 \\ .28 \\ .63 \\ .95 \\ 1.69 \\ .22 \\ .33 \\ .36 \\ 1.01 \\ .56 \\ .88 \\ .75 \\ .56 \\ 1.10 \\ .18 \\ \underline{.56} \end{array}$$

(34)
$\frac{1}{6} + \frac{3}{8} =$

(35)
2 ft. 6 in.
3 ft. 5 in.
4 ft. 9 in.

(36)
2 yr. 5 mo.
3 yr. 6 mo.
4 yr. 9 mo.
5 yr. 2 mo.
6 yr. 7 mo.

(37)
$$\begin{array}{r} 16\frac{1}{3} \\ 12\frac{1}{6} \\ 21\frac{1}{2} \\ \underline{32\frac{3}{4}} \end{array}$$

(38)
$25.091 + 100.4 + 25 + 98.28 + 19.3614 =$

Scales are published in various forms. Some of them, such as the Woody Arithmetic Scales, have a separate test for each of the four fundamental operations. They combine examples in one operation in whole numbers, fractions, decimals, and denominate numbers in the same exercise. Other scales, such as the Woody-McCall Mixed Fundamentals, the Stanford Achievement Test, the Los Angeles Diagnostic Tests in Arithmetic, and the Woody-VanWagenen Mixed Fundamentals, include examples in all processes in one test, arranged in the order of their difficulty. The same scale is given to pupils in all grades and consequently must cover a wide range of difficulty.

The score of the pupil is usually determined by the number of examples in the scale he can work correctly. This is an index of the highest level of difficulty of the examples that he is barely able to work correctly. These scores can be converted into arithmetic ages and scores, and used in other ways that are described in detail in the manuals for each of the tests. Scores can be related to intelligence by means of the achievement quotient technique. The data secured make it possible to compare the achievement of a class or of an individual with standard scores and to find whether the work is at the level to be expected of pupils of their grade or age. For example, note the scores below made on the Woody-McCall Mixed Fundamentals Test by pupils in a certain school and also the standard scores for each grade, which are expressed in terms of the average number of examples worked correctly by pupils in each grade in October.

TABLE I.

COMPARISON OF SCORES ON WOODY-MCCALL MIXED FUNDAMENTALS TEST WITH GRADE STANDARDS

GRADE	3	4	5	6	7	8
Standard Score.....	6.8	13.1	17.8	22.5	25.9	27.8
School B...........	6.4	11.7	17.2	24.3	26.7	28.4

In Table I, the class averages are somewhat below the standard scores in grades 3, 4, and 5, and slightly exceed the standards in grades 6, 7, and 8. This information gives the teacher comparative data of considerable value which in some schools are made the basis for grouping or classifying pupils. However, it furnishes no precise data as to the exact phases or processes of arithmetic in which the class may be deficient or as to the reasons for the deficiency of any particular individual. There is the

same limitation to the information supplied by a knowledge of the pupil's arithmetic age as there is to data concerning the number of examples worked correctly. For example, a teacher may learn that a sixth-grade pupil has ability equal to that of a normal fifth-grade pupil or that a 14-year-old pupil is achieving only as much as the typical 12-year-old pupil; still he has no definite information as to the specific nature of this pupil's arithmetic deficiency on the basis of which to provide the needed remedial work. The same limitation applies to the problem scales of Buckingham. They provide a means for measuring the ability of pupils to solve problems, but they supply no information as to the reasons for the failure of pupils to achieve as much as is expected of pupils of their level of progress. Monroe attempted to overcome this difficulty by providing a method of scoring the work of the pupil for both computation and the ability to determine the principles involved in the problem.

The value of these arithmetic scales is further limited by the fact that for most of them there is not a sufficient number of equivalent forms of each test to make possible repeated measurement at regular intervals during the year. Present practice consists chiefly of giving a rate test or a problem scale, either at the beginning or at the end of the year and then comparing the scores on the test with standards in order to evaluate the progress that has been made. What is needed is a series of tests or problem scales for each grade which would make possible a continuous inventory of the progress being made at regular intervals. The tests could be given at least once a month throughout the school year, so that the necessary teaching adjustments could be made at any time during the year.

(3) Curriculum tests in arithmetic. In order to make it possible to conduct systematic surveys of the work in arithmetic throughout the year, a series of standardized curriculum tests in arithmetic has been prepared by Brueckner.[1] There are ten tests for each grade, one for each month in the school year. The use of the name, "Curriculum Tests," is due to the manner in which the contents of each of the tests were determined. The procedure used was briefly as follows: It is possible to list in detail the specific abilities and arithmetic processes that are to be taught in each grade. At the beginning of the year the work usually consists largely of a review of the processes developed in the preceding year, few new processes being presented. In some months more new steps may be presented than in others because of differences in difficulty. This list of skills and abilities can be divided into sections, each of which covers roughly the work for one month, that is, the curriculum for the month. In constructing the Curriculum Test in Arithmetic for any one month, the first step was to include in the test the new skills and processes which would normally be taught in that month. When the number of examples secured in this way was inadequate, typical difficult examples from the work in previous months were added until the desired number of examples, graded according to approximate degrees of difficulty, had been assembled. In this way ten tests for each grade were prepared. A typical curriculum test for grade 3 is given on page 33. Similar tests for grades 4 to 8 are available. Obviously their content becomes increasingly difficult from grade to grade.

[1] L. J. Brueckner, *Curriculum Tests in Arithmetic Processes* (Philadelphia: The John C. Winston Company), Grades 3, 4, 5, 6, 7, and 8.

NAME.................... GRADE....... ROOM.......

CURRICULUM TEST III

DIRECTIONS: You will be allowed 10 minutes to work the examples below. Do your work on the paper in the space for each example. Work rapidly, but try to have every example correct. Begin to work when your teacher says "Start."

Number correct...	....
Rating...........	....

Find your rating and continue your *Progress Chart.*

1. $\begin{array}{r} 2 \\ +27 \\ \hline \end{array}$	2. $2 \times 2 =$	3. $\begin{array}{r} 26 \\ 29 \\ 18 \\ \hline \end{array}$	4. $\begin{array}{r} 92 \\ +9 \\ \hline \end{array}$
5. $\begin{array}{r} 95 \\ -28 \\ \hline \end{array}$	6. $\begin{array}{r} 80 \\ \times 3 \\ \hline \end{array}$	7. $\begin{array}{r} 168 \\ -94 \\ \hline \end{array}$	8. $\begin{array}{r} 152 \\ +286 \\ \hline \end{array}$
9. $\frac{1}{2}$ of 10 =	10. $2\overline{)64}$	11. $\begin{array}{r} 73 \\ -64 \\ \hline \end{array}$	12. $8 \div 2 =$
13. $\begin{array}{r} \$7.50 \\ +2.18 \\ \hline \end{array}$	14. $\begin{array}{r} 708 \\ -495 \\ \hline \end{array}$	15. $\begin{array}{r} 94 \\ 78 \\ 89 \\ 59 \\ \hline \end{array}$	16. $2\overline{)608}$

STANDARDS

Rating...........	1	2	3	4	5	6	7	8
Number correct...	0–1	2–3	4–6	7–8	9–11	12–13	14–15	16

The basis for the standards for each test is the scores made by pupils on that test when given at the end of the month for which it is constructed; for example, the standards in Curriculum Test I in each grade are based on scores made at the end of the first month by pupils in

CURRICULUM TESTS IN ARITHMETIC
Class Summary Sheet

SCHOOL, Longfellow GRADE ROOM 206 TEST III
DATE, December, 19— TEACHER, Mary ——

SUMMARY OF STANDARDS	Number of Pupils Making Each Standard	SUMMARY OF ERRORS ON EACH EXAMPLE	Number of Pupils Making Errors on Each Example
9	1	1	2
8	2	2	3
7	3	3	3
6	5	4	5
5	9	5	5
4	6	6	7
3	3	7	9
2	2	8	11
1		9	11
	—	10	9
Total number	31	11	14
Mid-standard	5	12	12
		13	11
		14	16
		15	12
		16	17
Comments: Much absence		17	19
due to influenza.		18	20

Recommendations: Give special work on zero difficulties in short division.

the first semester of that grade; the standards for Test II on scores made at the end of the second month of the first semester. The standards for each of the other tests are based on scores obtained in a similar manner. The standards for each test are given at the foot of the test sheet. These standards make it possible to determine each month whether a class or a pupil is at, above, or below the expected median score for pupils of that grade. They provide a continuous survey of the results on the basis of which necessary adjustments of instruction can be made in reviewing arithmetic processes, and in the teaching of the new steps.

NAME, Harry Nelson GRADE 3 ROOM 306

CURRICULUM TESTS IN ARITHMETIC

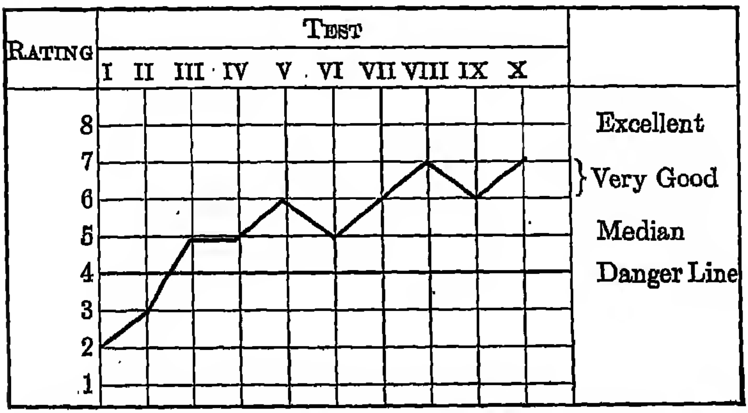
RECORD OF TEST RESULTS

Enter the results of the tests in the spaces below. Keep your record up to date. Write neatly and clearly.

NUMBER OF TEST	DATE TAKEN	NUMBER CORRECT	RATING
1....................	Sept. 18	7	2
2....................	Oct. 16	9	3
3....................	Nov. 13	10	5
4....................	Dec. 12	12	5
5....................	Jan. 24	14	6
6....................	Feb. 22	13	5
7....................	Mar. 22	14	6
8....................	Apr. 20	15	7
9....................	May 18	14	6
10....................	June 18	15	7

PROGRESS CHART

Use a chart similar to this chart to graph your scores. Watch your progress. Your teacher will show you how to make your graph.



A Typical Individual Record and Graph of Scores on Curriculum Tests.

The records of test scores constitute the basis of legitimate incentives to stimulate efforts on the part of the pupils to strive to increase their ability in arithmetic. An analysis of the work on the examples in the test also supplies valuable information as to the phases of the work on which additional practice is needed by the class or by individual pupils. (See page 35.)

Graphs showing the progress made by pupils and classes month by month can also be prepared on the basis of the results. (See page 35.).

Curriculum tests such as those that have just been described constitute an important part of the technical equipment of the teacher and the supervisor of arithmetic. They supply much more detailed information concerning the progress being made by the pupils in a class than can be secured by the arithmetic scales which are now being used, since no attempt is made in these scales to evaluate or measure the ability of the pupil to work examples of as wide a variety as are included in the curriculum tests.

A complete series of ten monthly standardized curriculum tests for each grade from 3 to 8 is available. The tests for each grade are based on the detailed work for that grade, as found in the Triangle Arithmetics, published by The John C. Winston Company.

Obviously, a single scale for grades 3 to 8, such as that of Woody, must contain a series of examples covering a much wider range of skills or processes than would be found in a curriculum test designed to cover the work for one month and for the period immediately preceding it. In addition to the fact that the curriculum tests are better adapted to measuring the processes in a grade than the scales, there is the advantage that they make it

possible to secure ten different comparable measurements of the status of the work in arithmetic at regular intervals during the year. Such information cannot be secured by means of the standard tests previously available because none of them were constructed so as to follow a specified curriculum, and because of the few forms of each test available.

(b) **Diagnostic tests.** Rate tests, scales, and curriculum tests supply much valuable information as to the relative status of the achievements of pupils in arithmetic. Still it has been found necessary to supplement them with analytical diagnostic tests which will assist in the location of the exact nature of the deficiency or difficulty that is preventing desirable growth in ability on the part of certain pupils.

One of the earlier forms of diagnostic tests, the Monroe Diagnostic Tests in Arithmetic, consists of a series of 21 rate tests, each of which measures the ability of the pupil in a particular phase of arithmetic. Illustrations of the types of examples included in the tests are given on page 38. The pupil's weakness is discovered by comparing his rating on each of the tests with standard scores and by locating the elements in which his score is below standard. These diagnostic tests supply important information concerning the skills which they measure, but they do not measure the ability of the pupil to work the many types of examples not found in the series of tests. Nor does the knowledge that a pupil *is* below standard on any one of the tests show *why* he is deficient in that particular phase of arithmetic. Additional information based on a detailed diagnosis is needed as to the specific difficulty or groups of difficulties which are the causes of his deficiency.

TYPICAL EXAMPLES FROM MONROE DIAGNOSTIC TESTS

ADDITION

TEST I	TEST II	TEST VII	TEST XII	TEST XV
4	7862	7	$\frac{1}{6} + \frac{1}{4} =$	$\frac{1}{6} + \frac{2}{8} =$
7	5018	6	$\frac{4}{8} + \frac{1}{7} =$	$\frac{3}{12} + \frac{5}{8} =$
2	1761	6	$\frac{3}{10} + \frac{3}{5} =$	
	5872	5	$\frac{2}{9} + \frac{[illegible]}{8} =$	
	3789	0		
		5		
		1		
		8		
		5		
		8		
		3		
		1		
		2		

SUBTRACTION

TEST II		TEST IX		TEST XIII
87	94	739	1858	$\frac{3}{4} - \frac{2}{5} =$
5	8	367	948	$\frac{4}{5} - \frac{3}{4} =$

MULTIPLICATION

TEST III	TEST VIII	TEST X			
6572	4857	560	807	617	840
6	36	37	59	508	80

TEST XIV	TEST XVIII		TEST XX	
$\frac{2}{5} \times \frac{3}{4} =$	657.2	67.50	487.5	57.28
$\frac{2}{5} \times \frac{3}{7} =$	.7	.03	.62	9.5
	46004	20250	302250	544160

(Pupil inserts decimal points in tests XVIII and XX.)

DIVISION

TEST IV	TEST VI	TEST XI	TEST XVI
8)3840	82)3854	47)27589	$\frac{2}{5} \div \frac{1}{3} =$

Test XIX	Test XVII	Test XXI
$.4\overline{)148}$ Ans.: 37	$.03\overline{)16.2}$ Ans.: 54	$.47\overline{)2758.9}$ Ans.: 587
$.9\overline{)65.7}$ Ans.: 73	$.07\overline{)1.82}$ Ans.: 26	$8.2\overline{)38.54}$ Ans.: 47

(In tests XVII, XIX, and XXI pupils write in answers and place decimal points.)

Diagnostic tests may be divided into three groups: 1, the sampling test, by means of which the teacher can determine the ability of the pupil to work selected examples of the types included in the test; 2, tests which seek to determine the specific elements in a process which may be the cause of the pupil's difficulty; and 3, tests which seek to determine the ability of the pupil to work examples of varied types, each type representing a different combination of the various skills that are the basis of tests of the second type. Tests of the third type inventory the complete area of the process being tested.

(1) The sampling test. In a sampling test the teacher or supervisor includes a selected variety of examples in different processes. After the test has been completed the work on the papers of the pupils is analyzed and the examples which are worked incorrectly most frequently, are determined. A typical sampling test is given on page 40. The curriculum test on page 33 may also be used as a sampling test, since, after it has been given the number of times that each example on the test was worked incorrectly can easily be determined.

Diagnostic Test VII[1]

This exercise will help you to locate your difficulties in working examples in fractions and decimals.

[1] Triangle Arithmetics, Book 2, Part I, page 250.

1. Add $4\frac{1}{4}$, $5\frac{5}{8}$, and $7\frac{7}{8}$.
2. Subtract $6\frac{7}{8}$ from $15\frac{1}{4}$.
3. $3\frac{1}{2} \times 1\frac{1}{4} \times 4$.
4. Divide $7\frac{1}{2}$ by 6.
5. Divide $3\frac{3}{4}$ by $7\frac{1}{2}$.
6. Find the cost of 24 inches of ribbon at \$.36 a yard.
7. Add 7.4, 8.25, and 563.
8. Subtract 7.5 from 18.25.
9. Multiply 27.5 by 16.
10. Divide 28.8 by 6.
11. Divide 753 by 16. Carry the work to two places.
12. Express .75 as a common fraction.
13. Write as words: 26.07; 18.206; 728.01.
14. Multiply \$.26 by 100; by 1,000; by 10.
15. Write a fraction and name its terms.
16. Add \$37.64, \$.04, \$7, and \$128.65.
17. Subtract \$4.68 from \$300.
18. Multiply \$76.43 by 800.
19. Divide 570,665 by 95.
20. At 25 cents a dozen, how much do 8 cookies cost?
21. Which is more, $\frac{7}{8}$ yard or $\frac{9}{10}$ yard?
22. Express 45 minutes as a fraction of an hour.
23. Find the cost of 2 pounds 4 ounces of meat at 40¢ a pound.
24. Draw a plan of a garden 100 feet square to the scale of 1 inch for 25 feet.
25. Find the volume of a trunk, $3\frac{1}{3}$ feet by $2\frac{1}{2}$ feet by $2\frac{2}{3}$ feet.

A blank, such as the one on page 34, can be used to show the complete distribution of errors on the various examples among the pupils in the class, and also the number of times each example was missed.

The use of the sampling test makes it possible to devise informal methods of locating the deficiencies in the work done by a class, since the purpose of such a diagnostic test is not to compare achievements with standards but to determine the ability of the pupils to work certain

types of examples. Sampling tests are not timed because their purpose is not to determine the number of examples of a given type the pupils can work in a given time, but to determine whether they are even able to work examples in the test.

The contents of sampling tests can be selected by applying the same principles that were used in the construction of the Curriculum Tests, previously described. The examples that make up the sampling tests can be selected from among those that are contained in a complete list of the specific skills and types of examples that constitute the curriculum. By a careful distribution of the various types of examples in the sampling tests and by varying the contents of such tests, a complete inventory can be made in a short time, of the processes that have been taught. Thus an excellent type of review is also provided.

The new types of objective examinations may be used as sampling tests. They enable the teacher to determine the types of information the pupils may lack; to determine their ability to select correct solutions to examples; to determine their grasp of important principles involved in problem solving; and to secure other kinds of valuable information. A typical exercise of this kind is the following completion test. Other types of useful informal tests are described in Chapter IX.

A Reading Test[1]

Supply the missing word in each of these sentences.

1. $\frac{3}{4}$ is a —— fraction.
2. $\frac{10}{9}$ is an —— fraction.
3. $1\frac{1}{2}$ is a —— number.

[1] Triangle Arithmetics, Book II, Part II, page 16.

4. 4 is a —— number.
5. $\frac{1}{4}$ and $\frac{3}{4}$ are —— fractions.
6. $\frac{1}{4}$ and $\frac{1}{2}$ are —— fractions.
7. When $\frac{1}{2}$ is changed to $\frac{4}{8}$ it is changed to —— terms.
8. When $\frac{6}{8}$ is changed to $\frac{3}{4}$ it is reduced to —— terms.
9. Before $\frac{1}{2}$ and $\frac{1}{4}$ can be added, they must be changed to a —— denominator.
10. To change $\frac{1}{2}$ to $\frac{4}{8}$, multiply each term by ——.
11. To reduce $\frac{8}{10}$ to $\frac{4}{5}$, —— each term by 2.
12. Inverting $\frac{3}{5}$ gives ——.
13. Inverting $2\frac{2}{3}$ gives ——.
14. Inverting 5 gives ——.
15. Before unlike fractions can be added or subtracted, they must be changed to —— fractions.
16. When changed to an improper fraction, $3\frac{7}{8}$ is equal to ——.
17. $\div \frac{5}{6}$ means the same as $\times$ ——.
18. A common denominator for $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ is ——.
19. Give the steps in working this example:
    $16\frac{1}{8} - 3\frac{1}{2} = ?$
20. Express the following in the simplest form:
    $6\frac{6}{6}$ $\quad 9\frac{4}{2}$ $\quad 12\frac{8}{6}$ $\quad 16\frac{20}{10}$ $\quad 18\frac{14}{12}$

Sampling tests enable the teacher to determine the types of examples with which an individual or a class has difficulty, but the exact reason for the difficulty can be discovered only by a careful analysis of the work of the pupil on the test paper. Such an analysis should reveal whether the example was worked incorrectly because of failure to understand the process involved, because of inaccuracy in computation, confusion of processes, faulty methods of work, or for other reasons. Such an analysis is greatly facilitated by applying the techniques described in succeeding chapters and by a knowledge of the most common kinds of mistakes made by pupils, lists of which are also given.

(2) Tests for determining the element in a process

which is the source of the deficiency. In this group there are two types: (a) those which aid in the selection of the specific elements in a process on which the pupil may be deficient; (b) those which show the *level* at which pupil mastery in a process breaks down.

(a) *Determining the element causing difficulty.*

In the primary grades it is important for the teacher to discover what specific number combinations the pupil does not know, since lack of knowledge of basic number facts is one of the chief causes of failure in the various processes in the intermediate grades. For this purpose the combinations may be grouped into sets according to their difficulty as shown in Chapter IV.

When pupils are tested on one group after the other, specific combination deficiencies can be discovered which may be made the basis of the remedial work in subsequent practice periods, each pupil practicing on the combinations he does not know. The Wisconsin Inventory Tests and Lunceford Diagnostic Tests are tests of this type.

Analysis of the example at the left in subtraction of fractions shows that certain elements are involved in working it, each of which may be the cause of difficulty. These elements are as follows:

$$\begin{array}{rcccccc} & 5\frac{1}{2} & = & 5\frac{3}{6} & = & 4\frac{9}{6} & \\ - & \underline{1\frac{5}{6}} & = & 1\frac{5}{6} & = & \underline{1\frac{5}{6}} & \\ & & & & & 3\frac{4}{6} & = 3\frac{2}{3} \end{array}$$

1. Ability to change fractions to a common denominator.
2. Ability to see that borrowing is involved in working the example.
3. Ability to change the form of $5\frac{3}{6}$ to $4\frac{9}{6}$.
4. Ability to subtract mixed numbers containing like fractions.
5. Knowledge that the answer must be expressed in lowest terms.
6. Ability to reduce the fraction in $3\frac{4}{6}$ to lowest terms.

Inability of the pupil to perform the work involved in any one of these six elements, assuming that his knowledge of the basic subtraction facts is adequate, may be the cause of failure to work the example correctly. To determine the exact element which is causing the difficulty the teacher can prepare an informal exercise of the following type which tests the pupil's ability to do the work involved in each element:

(*a*) Change the unlike fractions in the following pairs to like fractions:

1. $\frac{1}{2} = \frac{\ }{4}$, $\frac{3}{4} = —$
2. $\frac{1}{3} = —$, $\frac{3}{4} = —$
3. $\frac{1}{6} = —$, $\frac{1}{4} = —$
4. $\frac{1}{3} = —$, $\frac{7}{12} = —$
5. $4\frac{1}{2} = 4 —$, $2\frac{3}{4} = 2 —$
6. $5\frac{1}{3} = 5 —$, $2\frac{3}{4} = 2 —$
7. $6\frac{1}{4} = 6 —$, $2\frac{3}{8} = 2 —$

(*b*) In which of the following examples will it be necessary to borrow before you can subtract the numbers?

1. $6\frac{1}{4} - 4\frac{3}{4}$
2. $7\frac{1}{2} - 2\frac{1}{4}$
3. $6\frac{1}{3} - 2\frac{3}{4}$
4. $5\frac{5}{8} - 2\frac{1}{8}$
5. $9\frac{1}{3} - 6\frac{1}{3}$

(*c*) Supply the missing numbers in the following examples:

1. $1\frac{1}{4} = \frac{\ }{4}$
2. $1\frac{1}{2} = \frac{\ }{4}$
3. $1\frac{1}{3} = \frac{\ }{12}$
4. $8\frac{1}{2} = 7\frac{\ }{2}$
5. $7\frac{1}{3} = 6\frac{\ }{12}$
6. $7\frac{1}{4} = 6\frac{\ }{4}$; $-2\frac{3}{4} = 2\frac{3}{4}$
7. $7\frac{1}{2} = 7\frac{2}{4} = 6\frac{\ }{4}$; $-5\frac{3}{4} = 5\frac{3}{4} = 5\frac{\ }{4}$
8. $7\frac{1}{3} = 7 — = 6$; $-5\frac{3}{4} = 5 — = 5$

(*d*) Reduce the fractions in the following to lowest terms where they are not already so expressed:

1. $5\frac{2}{4} =$
2. $7\frac{12}{18} =$
3. $8\frac{9}{12} =$
4. $9\frac{11}{18} =$
5. $6\frac{8}{10} =$

(*e*) Subtract the following, expressing all fractions in answers in lowest terms.

$$1.\ \begin{array}{r} 9\frac{1}{4} \\ -\ 7\frac{3}{4} \\ \hline \end{array} \qquad 2.\ \begin{array}{r} 6\frac{1}{3} \\ -\ 4\frac{2}{3} \\ \hline \end{array} \qquad 3.\ \begin{array}{r} 8\frac{3}{8} \\ -\ 8\frac{5}{8} \\ \hline \end{array}$$

$$4.\ \begin{array}{r} 9\frac{3}{5} \\ -\ \frac{4}{5} \\ \hline \end{array} \qquad 5.\ \begin{array}{r} 7\frac{5}{12} \\ -\ 4\frac{11}{12} \\ \hline \end{array}$$

By means of such informal exercises as these the specific element in the subtraction process which is causing difficulty can be located. Such detailed analyses probably are needed only for pupils who have marked weaknesses in a particular process or for types of examples such as the following which contain special elements of considerable difficulty:

$$1.\ \begin{array}{r} 9 \\ -\ 6\frac{2}{3} \\ \hline \end{array} \qquad 2.\ \begin{array}{r} 4\frac{3}{4} \\ +\ 9\frac{5}{6} \\ \hline \end{array} \qquad 3.\ 8\frac{1}{2} \times \frac{4}{5} = \qquad 4.\ 7\frac{1}{2} \div 5 =$$

$$5.\ \begin{array}{r} .7 \\ .6 \\ .9 \\ \hline \end{array} \qquad 6.\ \begin{array}{r} 265 \\ \times\ 408 \\ \hline \end{array} \qquad 7.\ 74\overline{)22{,}363} \qquad 8.\ 2.7\overline{)685}$$

The great variety of types of examples in each process which involves different combinations of skills and each of which consequently contains different possible causes of difficulty is a factor which must be taken into consideration in the development of such diagnostic exercises as that presented above for the subtraction example. Such detailed analyses of the elements involved in working any example make clear to the teacher the variety of steps that must be presented in teaching the method of working the example. The labor involved in preparing suitable diagnostic exercises of this type can be considerably reduced by limiting the analyses to the difficult types of examples in the various processes or by using the ready-made exercises found in the newer textbooks and in sets of remedial exercises.

The application of this technique of diagnosis involves first the location of the types of examples with which a pupil or a class has difficulty by means of a sampling test. When the difficult spots have been determined the next step is to give each pupil the exercise which will aid in the location of the exact element in that type of example which is giving him difficulty. Usually it will be necessary to use such analytical materials with only a small group of pupils who do not appear to profit from the help given by the teacher in the class presentation. Such exercises are not timed nor are standards available which can be used for comparative purposes. The function of such tests is to aid the teacher to locate the element in examples of a given type which is the cause of difficulty, not to measure the ability of the pupils to work such examples.

(b) *Determining the level at which mastery breaks down.*

It has been suggested that it is possible to determine the *level* at which pupil mastery of a process breaks down. In order to apply this method, the important skills in arithmetic are first analyzed into a number of component parts, each of which can be measured in isolation. The Compass Diagnostic Tests in Arithmetic are based on this principle. The diagnostic test in addition of whole numbers which is the first one in this series of twenty tests is given on pages 47 to 49. The test consists of five parts, as follows:

	Time Allowed
Part 1, The Basic Addition Facts..........	3 minutes
Part 2, Higher Decade Addition..........	6 minutes
Part 3, Column Addition.................	3 minutes
Part 4, Carrying in Column Addition......	6 minutes
Part 5, Checking Answers in Addition.....	9 minutes

COMPASS DIAGNOSTIC TESTS IN ARITHMETIC

RUCH—KNIGHT—GREENE—STUDEBAKER

EDITED BY G. W. MYERS

TEST I: ADDITION OF WHOLE NUMBERS: FORM A

Name.................................Grade.....................Boy or girl?............

Age....When is your next birthday?......How old will you be then?....

School..Date.........................
(Name) (City) (State)

SUMMARY OF PUPIL'S SCORE	PART 1	PART 2	PART 3	PART 4	PART 5	TOTAL
Scores on Parts of Test						
Educational Age Equivalent						
Grade Equivalent of Score						

PART 1—BASIC ADDITION FACTS

Add:

0 + 2 =	4 + 1 =
1 + 7 =	5 + 3 =
2 + 2 =	1 + 0 =
7 + 6 =	6 + 4 =
2 + 5 =	3 + 4 =
2 + 1 =	2 + 7 =
0 + 9 =	1 + 6 =
9 + 2 =	3 + 3 =
3 + 9 =	7 + 4 =
5 + 8 =	0 + 0 =
9 + 1 =	9 + 9 =
8 + 5 =	7 + 5 =
2 + 8 =	9 + 4 =
8 + 3 =	7 + 2 =

0	9	5	2	3	1	7
2	1	0	3	4	1	5
9	8	5	7	6	4	2
2	4	4	4	2	1	6
9	4	8	0	4	2	6
5	0	6	7	2	5	6
5	2	4	3	7	8	8
3	4	5	7	3	3	8
9	0	1	9	2	7	1
6	9	7	7	8	6	8
7	1	7	5	8	3	6
8	6	7	9	0	9	3

Score on Part 1 = Number right = ...

[Total possible score = 70 points]

PART 2—HIGHER DECADE ADDITION

Add:

2	1	6	8	18	4	9	7	24	10	6	16	14	0
20	21	22	13	6	12	29	25	7	2	12	6	7	10

4	24	21	4	32	1	4	38	4	33	7	1	11	5
42	2	4	10	6	18	16	7	20	3	13	24	9	10

15	0	1	33	40	2	1	5	31	6	9	2	24	19
9	12	11	5	2	25	14	28	9	11	20	36	9	4

8	9	32	10	3	7	0	2	6	7	19	9
16	27	2	7	18	26	31	26	13	12	2	23

Add:

$19 + 7 =$	$14 + 2 =$	$10 + 3 =$
$12 + 2 =$	$11 + 7 =$	$18 + 8 =$
$20 + 7 =$	$28 + 7 =$	$0 + 20 =$
$26 + 6 =$	$18 + 7 =$	$22 + 0 =$

Score on Part 2 = Number right = ...

[Total possible score = 66 points]

PART 3—COLUMN ADDITION

Add:

4	7	6	9	6	9	9	4	7	8	2	6
7	4	6	4	4	7	2	2	6	0	4	3
5	9	9	6	0	8	4	9	7	1	3	5
	1	3	4	7	9	7	7	4	2	8	7
		4	9	5	1	5	8	8	4	1	0
				0	6	9	8	7	1	8	4
							8	6	2	6	7

Score on Part 3 = Number right × 5 = ...

[Total possible score = 60 points]

PART 4—CARRYING IN COLUMN ADDITION

Add:

1.	2.	3.	4.	5.	6.	7.
29	98	17	47	117	766	27
47	13	13	4	960	982	96
		67	13	517	932	77
			49		448	84
			29			98
						17
						26

8.	9.	10.	11.	12.	13.
9132	7027	162	653		
5112	249	4914	5156		
2348	7191	9737	7360		
7417	9235	67	2661		
	9059	9280	4099		
		427	9730		
			9222		

12. Copy and add: 62 + 604 + 827 + 797 + 997 = ? (Put your work under 12 above.)

13. Copy and add: 76, 64, 60, 97, 35, and 20. (Put your work under 13 above.)

Score on Part 4 = Number right × 10 = ...

[Total possible score = 130 points]

PART 5—CHECKING ANSWERS IN ADDITION

Directions: Some of the printed answers below are right, and some are wrong. Check each sum by adding downwards. Write your check answer on the line at the top of the example. The first one is already done correctly.

797							
218	165	6566	9270	7774	887	162	653
128	923	3157	9984	9447	756	4914	5156
456	928	4918	4763	7267	798	9737	7360
787	466	7194	7163	2848	91	67	2661
	2484	77	9650	9192	700	9280	4099
		21792	3083	36528	206	427	9730
					3618	24587	9222
							38861

Score on Part 5 = Number right × 10 = ...

[Total possible score = 70 points]

The five tests are given consecutively, the pupils being allowed the time indicated for work on each part. Their ability is measured by the score they make in each part of the test, based on the number of examples worked correctly. No attention is paid to the number of examples worked incorrectly. Each part of the test is essentially a rate test. Standard scores in terms of the number of examples are provided in each part of the test, which make it possible to compare the scores made by each pupil with reasonable norms. Scores may be expressed as arithmetic ages for each part of the test and for the whole. If the pupil's score is below the standard on some part or parts of the test, it is assumed that he is deficient in the skills contained therein, thereby locating the level at which his mastery of the process breaks down. A typical record is given in Figure 1 for Test III, Multiplication of Whole Numbers.

NAME, Elsie Schmidt GRADE L4 BOY OR GIRL: Girl

AGE 10. WHEN IS YOUR NEXT BIRTHDAY? September 4.

HOW OLD WILL YOU BE THEN? 11

SCHOOL, Barton. (Name) Minneapolis, (City) Minn. (State) DATE, May 8, 1928.

SUMMARY OF SCORE	PARTS I	II	III	IV	V	VI	TOTAL
Number of examples correct	49	53	42	29	1	5	179
Educational age equivalent.	..	..	..	..	..	..	9–6
Grade equivalent of scores..	H4	L4	L4	L4	H3	H3	L4

FIGURE 1.

The pupil whose record is shown is in the low fourth grade. Her score on Part I, Basic Multiplication Facts,

is equivalent to the High 4 standard. Her scores on Part II, Additions Needed in Multiplication in Parts 4, 5, and 6; Part III, Carrying in Addition used in Multiplication, and Part IV, Fundamentals in Multiplication, are equivalent to the standards for the Low fourth grade, the grade she is in. Her scores on Part V, Checking Multiplication, and Part VI, Finding Errors in Multiplication, are both considerably below the standard for her grade. Remedial work must begin at the levels represented by the materials and skills for Parts V and VI.

Just what is meant by "level" has not been clearly defined by the authors of this series of tests. An analysis of some of the tests shows that many of the important types of examples in the process have been omitted. This probably is due to the fact that it is the theory of the authors that the ability of a pupil can be adequately determined by measuring his ability to work a satisfactory number of examples from a fairly large sampling of the total possible list of types of examples in the process rather than all types. These tests therefore do not form the basis of a complete diagnosis, nor do they provide techniques by means of which the *why* of failure at a particular level can be discovered. The fact that a pupil is below the standard on one part of the test, say in subtraction of fractions, may be due to marked inability to work examples involving zero difficulties or borrowing procedures, or to some difficulty on some specific skill lumped in the same test with other specific skills, and not to an all around weakness in subtraction of fractions.

No attention is paid to the number of examples attempted and therefore the relative accuracy of the pupil's work is not determined. Specific types of difficulties are not located by these tests, nor are faulty methods of work

revealed. If a pupil is assigned practice work on the basis of his test scores above, such exercises may tend further to establish, rather than to eliminate, faulty habits which cannot be detected by such tests.

(3) Tests which determine the ability of the pupil to work a wide variety of types of examples in each process. An analysis of the learning process in each of the operations in arithmetic shows that the pupil may encounter a wide variety of types of examples in each process. Sets of examples can be prepared, each of which represents a particular combination of specific skills and elements not found in the other examples in the set. This point has been made clear by Monroe[1] in the following statement:

"Assuming the possibility of this conclusion, a little consideration will reveal that there are a large number of types of examples in division of decimal fractions. Even within a range of relatively simple examples in division of decimals it is possible to enumerate a considerable number of types. If each type of example requires a specific ability, it is clearly very important that this field of subject-matter be carefully analyzed to determine the fundamental types of examples. It is also obvious that it is extremely important that each teacher be aware of the several types of examples which exist and make certain that each pupil is trained in handling each type of example."

When the pupil is learning to work examples in a process it is desirable, therefore, that the practice exercises used to develop skill contain these varied types of examples in order that the whole area of the process may be covered. For example, in considering the skills involved in the addition of proper fractions the following types must be considered:

[1] W. S. Monroe, "The Ability to Place the Decimal Point in Division," *Elementary School Journal*, Vol. 18, pp. 287–98.

ADDITIONAL TYPES

$$
\begin{array}{lllll}
1.\ \frac{1}{3} & 2.\ \frac{1}{4} & 3.\ \frac{3}{4} & 4.\ \frac{3}{5} & 5.\ \frac{5}{6} \\
\underline{\ \ \frac{1}{3}} & \underline{\ \ \frac{1}{4}} & \underline{\ \ \frac{1}{4}} & \underline{\ \ \frac{4}{5}} & \underline{\ \ \frac{5}{6}} \\
\ \ \frac{2}{3} & \ \ \frac{2}{4} = \frac{1}{2} & \ \ \frac{4}{4} = 1 & \ \ \frac{7}{5} = 1\frac{2}{5} & \ \ \frac{10}{6} = 1\frac{4}{6} = 1\frac{2}{3}
\end{array}
$$

In this group of addition examples, each pair of fractions has a common denominator. In example 1, the sum, when found, is already expressed in lowest terms; in example 2, the sum must be reduced to lowest terms. A new element is introduced in example 3, since the sum is expressed as an improper fraction which when reduced to simplest form is a whole number. In example 4, the sum is an improper fraction which is reduced to a mixed number, involving no further reduction. In example 5, the sum is an improper fraction which when reduced to a mixed number contains a proper fraction ($\frac{4}{6}$) that must be reduced to lowest terms. Each of these examples, therefore, involves a different combination of skills and must be considered as a separate type in the construction of drill materials or diagnostic exercises.

These types can be varied by introducing the element of unlike denominators which greatly increases the difficulty of each basic type and, therefore, practically results in a new series of types each of which may cause difficulty. Note the following addition examples paralleling the five given above in which the only new difficulty included is the factor of reducing the fractions to a common denominator.

$$
\begin{array}{lllll}
1.\ \frac{1}{3} & 2.\ \frac{1}{3} & 3.\ \frac{1}{2} & 4.\ \frac{3}{4} & 5.\ \frac{5}{6} \\
\underline{\ \ \frac{1}{4}} & \underline{\ \ \frac{1}{6}} & \underline{\ \ \frac{4}{8}} & \underline{\ \ \frac{2}{5}} & \underline{\ \ \frac{1}{2}} \\
\ \ \frac{7}{12} & \ \ \frac{3}{6} = \frac{1}{2} & \ \ \frac{8}{8} = 1 & \ \ \frac{23}{20} = 1\frac{3}{20} & \ \ \frac{8}{6} = 1\frac{2}{6} = 1\frac{1}{3}
\end{array}
$$

The six examples in division of proper fractions, given below, also contain elements which make each of them a specific type. In type 1, the quotient is a whole number

greater than unity; in type 6, the answer is unity. The answers for types 2 and 3 are fractions, reduction being involved in the latter type. In types 4 and 5, the

DIVISION TYPES

1. $\frac{1}{2} \div \frac{1}{4} = \frac{4}{2} = 2.$	4. $\frac{1}{6} \div \frac{1}{8} = \frac{8}{6} = 1\frac{2}{6} = 1\frac{1}{3}.$
2. $\frac{1}{3} \div \frac{1}{2} = \frac{2}{3}.$	5. $\frac{1}{5} \div \frac{1}{7} = \frac{7}{5} = 1\frac{2}{5}.$
3. $\frac{1}{8} \div \frac{1}{4} = \frac{4}{8} = \frac{1}{2}.$	6. $\frac{1}{8} \div \frac{1}{8} = \frac{8}{8} = 1.$

quotients are mixed numbers, type 4 involving reduction to lowest terms.

The types here given may be made more difficult by increasing the sizes of the numerators to numbers larger than 1. Complete analyses of the important types of examples in each of the processes in whole numbers and fractions are given in Chapters IV and V.

In analyzing the types of examples in decimals the chief difficulty involved, in addition to the computation, is the proper use of the decimal point in each process in various types of examples. The analysis of types of examples of decimals can be made on this basis. Such analyses of types of examples in decimals are given in Chapter VI.

In making a study of pupil difficulties in arithmetic, it is necessary to survey their abilities over a wide range of skills in each process and to determine their ability to work the numerous types of examples in which these skills may occur in different combinations, or in which specific difficulties, such as those involving zeros, are present. A complete diagnosis of deficiencies cannot be made on the basis of the work done by the pupil on a few examples in each process selected at random, since it is practically certain that through this procedure the ability of the pupil to work difficult types of examples

not included in the test cannot be determined and serious deficiencies may not be discovered.

The necessity of such a detailed analysis of types of examples has been recognized by Buswell and John.[1] In their study of the nature of the faults of pupils in work in the fundamental processes they used tests covering the entire range of types of examples. In the studies dealing with the most common sources of errors in fractions and decimals, that are reported in subsequent chapters, the surveys of errors were made on the basis of an analysis of the mistakes made on diagnostic tests made up of a wide variety of types of examples in each of the processes. Similar analyses of difficulties have previously been based on the errors made on a small number of examples covering only a small part of the total area of the processes in fractions.

Diagnostic tests of this type are not timed. Their function is to make it possible for the teacher to determine the specific types of examples in each process which the pupils may not be able to work, not to compare their rate of work or their accuracy with standard scores. A class summary (see page 56) may be made of the number of times each of the examples was worked incorrectly, thereby locating the places where remedial work is needed.

As has been indicated, the nature of the specific element which is the source of the difficulty must then be determined by a careful analysis of the pupil's work, or by means of the specific types of diagnostic tests previously described. In some cases a special study of the pupil's methods of work must be made by methods described in the discussion of individual diagnostic procedures.

[1] G. T. Buswell and L. John, "Diagnostic Studies in Arithmetic," *Supplementary Educational Monograph, No. 30* (Chicago, Illinois: University of Chicago Press, 1926).

BRUECKNER DIAGNOSTIC TEST IN FRACTIONS

Class Diagnostic Sheet

SCHOOL, Marshall GRADE 7B[1] ROOM 309

TEACHER, Florence Smith DATE, October 4, —

Record opposite each example the number of pupils that missed it. This analysis will reveal class weaknesses and the necessary remedial work. It should be followed by a study of individuals.

	Example	Addition	Subtraction	Multiplication	Division
Row I	1	6	3	12	19
	2	8	9	13	20
	3	9	8	11	22
	4	17	11	21	23
	5	14	8	20	22
Row II	1	3	6	18	15
	2	16	14	20	14
	3	6	6	12	11
	4	4	8	11	21
	5	7	8	16	23
Row III	1	11	10	18	17
	2	10	15	12	15
	3	15	10	7	23
	4	12	12	18	21
	5	15	14	14	..
Row IV	1	9	15	18	25
	2	19	16	16	20
	3	18	13	22	19
	4	11	14	23	23
	5	15	11	19	..
Row V	1	15	9	17	20
	2	15	14	20	22
	3	17	6	24	20
	4	15	15	23	22
	5	13	7	24	20
Row VI	1	9	13	20	19
	2	16	11	21	18
	3	12	10	22	23
	4	7	16	24	19
	5	21	15	16	..
Row VII	1	9	5	15	18
	2	16	7	20	22
	3	18	12	16	23
	4	14	11	24	27
	5	22	17	23	27
Row VIII	1	10	15	20	18
	2	17	11	15	20
	3	18	14	23	15
	4	19	15	26	25
	5	22	8	27	23
Row IX	1	..	17	17	..
	2	..	20	23	..
	3	..	18	26	..
	4	..	15	20	..
	5	..	23	22	..

REMARKS:

Diagnostic tests of this kind are also very useful in connection with the development of instructional materials.[1] The contents of the practice exercises can be constructed in such a way that the types of examples found in an analysis of possible types will all be included, and according to certain specifications set up in advance. In this way, the serious limitations that have been found to exist in the present-day practice exercises in the various processes can be overcome. The nature of these deficiencies is made clear by the following conclusions based on a study by Brueckner[2] of the distribution of practice on the various types of examples in the subtraction of fractions found in ten modern textbooks:

> The range in the amount of error on the 58 types of subtraction examples by a group of 167 sixth-grade pupils is from 7 per cent on type $(\frac{5}{8} - \frac{3}{8})$ to 40 per cent on type $(3\frac{1}{6} - 1\frac{2}{3})$. It is therefore obvious that each of these types should appear in drill materials in subtraction of fractions. Some of the types on which there were the largest amounts of error are entirely missing in some of the texts.
>
> Textbook writers apparently have not constructed their drills in the subtraction of fractions according to specifications which consider the question of types of example as a factor. In the ten textbooks analyzed, the range in the number of types missing was from 16 to 46.
>
> Very little practice is given in solving examples of the types that do occur in textbooks, since on the average 75 per cent of the types that are found in texts occur less than five times.
>
> There is no agreement as to what types of examples should occur in drills, there not being a single one of the 58 types that occurs in all ten textbooks.

[1] An excellent series of such diagnostic tests and practice exercises may be found in L. J. Brueckner, C. J. Anderson, G. O. Banting, and E. Merton, *Diagnostic Tests and Practice Exercises*, Grades 3, 4, 5, 6, 7, and 8 (Philadelphia, Pennsylvania: The John C. Winston Company, 1929).

[2] L. J. Brueckner, "A Technique for Analyzing the Distribution of Drill in Fractions," *Journal of Educational Method*, Vol. 7, pp. 352–58.

3. Factors to be Considered in Selecting a Test.

The most important consideration for the teacher to bear in mind in selecting a test is the purpose for which the test is to be given. If the teacher merely wishes to secure a picture of the present status of her class, a survey test should be given. If the teacher wishes to discover the sources of difficulty of pupils in the class, a diagnostic test should be given. When necessary the teacher should supplement the information secured by these tests by data of other types, that would help her to locate the causes of deficiency in the work of pupils whose scores are much below the standard for the grade.

Not all tests that are now available have the same value. This is due to the fact that they have not always been carefully constructed, and that basic principles that should underlie the construction of a test have not been taken into consideration. Briefly stated the criteria to be considered in the evaluation of a test are the following:

1. Ease of giving and scoring the test.

2. The validity of the test; that is, the general worthwhileness of the test, the extent to which it parallels the curriculum being taught, and the value of the test as a means of measuring what it purports to measure.

3. The reliability of the test; that is, the degree of accuracy of measurement, the stability of a measure of the ability of the pupil on various forms of the same test, and the degree to which the test measures what it is claimed to measure.

4. The cost and the availability of equivalent forms of the test. To allow for repeated measurement, tests with duplicate or equivalent forms should be selected.

Those who may be interested in making a detailed study of these factors will find an excellent discussion of criteria to be considered in evaluating standard and unstandardized tests in G. M. Ruch, *The Objective or New-Type Examination*, published by Scott Foresman and Company. Similar discussions are given in the texts listed in the bibliography at the end of this chapter. Manuals that are published for standard tests usually contain data regarding their reliability, their validity, the basis for the standards or norms, and other pertinent data.

Problems for Study, Reports, and Discussion

1. Show that arithmetic consists of a large number of specific skills and abilities.
2. Why is rate of work an important factor in ability in arithmetic? What provision is made in textbooks for giving standards for rate of work?
3. How does a pupil's rate of writing affect his scores in an arithmetic rate test? Why?
4. Is 100% a reasonable standard for accuracy in long column addition in grade 5?
5. How does a scale differ from a rate test? What aspect of ability is measured by a scale? What arithmetic scales are available?
6. What is meant by the "area" of a process?
7. Select from the tests listed in the appendix and those described in this chapter a group of tests and exercises that you feel should be part of the teaching equipment in arithmetic instruction.
8. Examine textbooks in arithmetic to find out what provision is made for survey tests, diagnostic tests, and practice exercises. How are standards stated?
9. Prepare a sampling test that may be used to determine the ability of the pupils to work examples in some phase of arithmetic processes. Give the test and determine the frequency with which pupils work each example incorrectly.

10. Analyze carefully the skills involved in the diagnostic exercise on page 44. Prepare a similar exercise. Give it to a class of pupils and analyze the results.

11. What is meant by a "type" example in fractions?

12. Why are analytical diagnostic tests not timed?

13. How may the validity of a test be determined?

14. Examine the manual of some standard test and find out what information it contains concerning the procedure followed in preparing and standardizing the test, its reliability, validity, norms, and method of scoring.

Selected Bibliography

In addition to the references cited in the previous discussion, the reader may refer to the following selected references for supplementary discussions on the uses of tests in the measurement of arithmetic ability.

The Appendix contains a complete list of arithmetic tests and instructional materials.

1. Buswell, G. T., and Judd, C. H., "Summary of Educational Investigations Related to Arithmetic," *Supplementary Educational Monograph*, No. 27 (Chicago: University of Chicago, 1925).

2. Gilliland, A. R., and Jordan, R. H., *Educational Measurements and the Classroom Teacher* (New York: The Century Company, 1925).

3. Greene, H. A., and Jorgenson, A., *The Use and Interpretation of Educational Tests* (New York: Longmans Green and Company, 1929).

4. Gregory, C. A., *Fundamentals of Educational Measurements* (Boston: Houghton Mifflin Company, 1923).

5. Hines, H. C., *Guide to Educational Measurements* (Boston: Houghton Mifflin Company, 1923).

6. McCall, W. A., *How to Measure in Education* (New York: The Macmillan Company, 1923).

7. Monroe, W. S., *Measuring the Results of Teaching* (Boston: Houghton Mifflin Company, 1918).

8. Monroe, W. S., DeVoss, J. C., and Kelly, F. J., *Educational Tests and Measurements* (Boston: Houghton Mifflin Company, 1924).

9. Paulu, E. M., *Diagnostic Testing and Remedial Teaching* (Boston: D. C. Heath and Company, 1924).

10. Trabue, M. R., *Measuring Results in Education* (New York: American Book Company, 1924).

11. Wilson, G. M., and Hoke, K. J., *How to Measure* (Revised) (New York: The Macmillan Company, 1928).

12. Woody, C., "Measurements of Some Achievements in Arithmetic," *Teachers College Contributions to Education*, No. 80 (New York: Columbia University, 1920).

CHAPTER III

TECHNIQUE OF INDIVIDUAL DIAGNOSIS

Survey and diagnostic tests give the teacher important information as to the pupil's ability in the various processes in arithmetic, and aid considerably in the location of the causes of weakness in some cases. But these tests must be supplemented by a special study of the work of an individual pupil who is much below the standard for his grade or is not making satisfactory progress due to some sort of hidden difficulty which is not apparent from a study of his work on the test. The number of pupils in need of such careful study will vary from class to class. Ordinarily not more than from five to ten in a class of forty require such an analytical examination.

Diagnostic procedures to be used with individual cases may be classified as of three types: 1, general, 2, analytical; and 3, psychological.

1. GENERAL DIAGNOSIS.

By means of survey tests and scales, the teacher can determine the general level of the pupil's ability in arithmetic. These data provide a basis of selecting pupils whose work is below the standard for their grade. As has been indicated, such scales and survey tests give almost no information as to the exact causes or nature of the deficiencies of pupils whose work is not up to standard. A graph such as that on page 107 may show that all classes in the school are above the average. The

giving of a survey test, however, is only the first step in an effective testing program. When a class score is considerably above average there is often great danger that the pupils who are at the lower end of the class distribution may be overlooked and no special adjustment made to bring their work up to the standard. For example, a certain principal in his report of the results of a testing program in arithmetic, which showed the class averages to be above standard throughout, stated that because of this fact drill in arithmetic need not be stressed, thereby implying that the work of all pupils was satisfactory. An analysis of the scores made by individuals in his classes showed that several of the pupils were considerably below standard and in need of special individual help.

When a class score is below the standard, possible factors which have produced this result should be investigated. The mental level of the class may be such that their achievement may really be exceptional for pupils of their mental level; the class may contain many average pupils; there may have been excessive absence for various reasons; there may have been a considerable turnover in the pupil population; the instructional materials may be unsatisfactory; the teacher may be incompetent; or the training that the pupils received in earlier grades may have been inferior.

When survey tests can be given only once during the year, they probably should be given early in the school year instead of at its end, as is often done, because then the data can be used as the basis of adjustment of instruction during the remainder of the year. About the end of the first month of school would seem to be the best time for these tests, since by then the "warming up

process" or review work which is necessary to overcome the retrogression in ability in arithmetic which occurs during the summer vacation is completed and a more reliable picture of the conditions would be secured than by a test given immediately after the opening of school. Obviously, it is desirable to repeat the testing program later in the year to secure data as to the growth that has taken place as a result of the work during the year. A class, which may be below standard at the end of the year, may actually have made excellent progress when their rating is compared with similar ratings secured earlier in the year. A series of standard tests given at more frequent intervals and standardized for the work being covered by the class provides a more adequate basis of supervision and instruction than two tests that may not even cover the curriculum that the pupils are following, given at the beginning and the end of the year.

2. Analytical Diagnosis by Means of Tests.

In this group may be placed all such diagnostic procedures involving the uses of tests as are employed to determine:

(a) the particular process in which the pupil is deficient;

(b) his ability to work certain types of examples contained in sampling tests;

(c) the particular element or skill in a process which may be the cause of the deficiency;

(d) the level at which pupil mastery of a process breaks down;

(e) his ability to work a large variety of examples in a process, such as are contained in comprehensive diagnostic tests.

As has been shown in the preceding chapter, there are available at the present time standard diagnostic tests which make it possible to determine the weaknesses of pupils by any of the five methods listed above. If a pupil's rating on a test, containing a partial sampling of the varieties of examples in a process, is considerably below standard there is every reason to believe that he may have difficulties in the process as a whole which should be investigated. Such sampling tests have the limitations that they do not show whether the pupil knows how to work examples of the type not included in the test, nor do they reveal the specific nature of the difficulty.

Standard diagnostic tests, or informal exercises, which assist in locating the particular element in an example or in a process that is the basis of the difficulty, are excellent teaching guides. They indicate the steps that must be presented in teaching the process. Such informal exercises can be prepared by any teacher who understands the technique of analyzing the skills involved in any given example, described in the preceding chapter.

The knowledge that a pupil is having difficulty because of some specific element, may be made the basis of the remedial teaching. Such remedial work is given after the teacher has discovered by some method whether the difficulty is due to lack of comprehension of the process, lack of some more basic skill, or to other causes. Diagnostic tests which enable the teacher to determine the level at which pupil mastery breaks down, or to ascertain which types of examples in a particular process a pupil is not able to work, are analytical in character and indicate the places where remedial work must be done. Neither of these types of tests indicates the specific causes of the difficulties either of individuals or of the class as a

whole. In order that the teaching may function effectively, it is important that the exact nature of the interfering element, process, skill, or method of work be determined. This is especially needful for pupils whose work shows little improvement.

3. Psychological Diagnosis.

When the teacher has discovered, either by the general or by the analytical methods of diagnosis, that the pupil is deficient in arithmetic, the problem then presents itself of how best to discover the exact cause or nature of the difficulty.

(a) *Specific procedures in psychological diagnosis.* The specific procedures used in the psychological type of diagnosis have been devised for the purpose of locating the exact causes of failure or difficulty in arithmetic. Such procedures supplement the data secured from survey and diagnostic tests. The two chief methods of psychological diagnosis are: 1. To analyze the written work of the pupil. 2. To observe the pupil's mental processes in working examples, which can be done when he is required to do all of the work aloud.

(1) *Analyzing the written work of the pupil.* Many difficulties of pupils are apparent after an analysis has been made of their written work. Some of these difficulties are easily removed by a few helpful suggestions.

(a) $\frac{1}{2} + \frac{1}{6} = \frac{2}{8} = \frac{1}{4}$ In example (a) at the left it is evident that the pupil added the two numerators and the two denominators to find the terms of their sum.

(b) $\begin{array}{r} 85 \\ -\ 39 \\ \hline 54 \end{array}$ In example (b) the pupil probably subtracted the 5 from the 9 instead of borrowing. Both of these examples illustrate faulty methods of work and show that some direct teaching is needed.

(c) $4 \times 36 = 162$ In example (c) the pupil carried the wrong number (4) from (24), the product of 4×6, and wrote the (2).

$$\text{(d)}\quad \begin{array}{r} 73 \\ +\,26 \\ \hline 98 \end{array}$$

In example (d) the pupil gave the incorrect sum of $6 + 3$. Both of these examples illustrate possible carelessness in work, or errors due to lapses in attention.

$$\text{(e)}\quad \begin{array}{r} 98 \\ 47 \\ 56 \\ \hline 191 \end{array}$$

In some cases it is more difficult to determine the exact cause of the error by an examination of the written work. In example (e) the pupil found the correct sum for the first column and the incorrect sum for the second. The error may have been made because he had difficulty in carrying or because of faulty sums in the several additions in that column.

(f) $1\frac{1}{3} \div 3\frac{1}{3} = \frac{1}{6}$ In example (f) it is not possible to determine the cause of the error from the work given. The pupil may merely have guessed an answer; he may have tried to work the example mentally and have omitted some important step; he may not have known how to reduce mixed numbers to improper fractions or how to invert the divisor. In order to discover the exact cause of the difficulty in examples, such as (e) and (f) in which the reasons for the errors are not obvious, the only satisfactory way to proceed is by means of a conference with the pupil in which his work is discussed and his difficulties are ascertained. While an analysis of the written work, in many cases, reveals the kinds of errors that were made, it does not give any information as to the mental processes of the pupil while working the examples, his methods of work, or his specific weaknesses.

(2) *Observing the mental processes and habits of work of the pupil.* While survey and diagnostic tests may give

the information that a pupil is considerably below a desirable standard in one or more processes or elements of processes, they yield no information as to his mental habits and methods of work. A pupil should not be assigned practice exercises in the process in which he is deficient before the teacher has made certain whether his difficulty may not be due to faulty, roundabout methods of work, lack of knowledge of the method of working the examples in the exercises, or some specific deficiency, such as inability to carry in addition or to borrow in subtraction. This can best be done by a careful observation and analysis of the work of the pupil as he works the examples orally.

One of those who first used this technique was Courtis[1] who describes methods of diagnosis and typical errors in processes in his Teacher's Manual for the Standard Practice Exercises in Arithmetic. Uhl[2] discussed the value of standardized material as the basis for diagnosis and described typical errors revealed by an analysis of the oral work of pupils in solving examples. The first blank for recording typical errors in arithmetic and the first standard test for analyzing pupil difficulties in the processes with integers were published in 1922.[3] These materials were the forerunners of the much more carefully constructed diagnostic tests and record blanks which are now used in diagnostic work in all arithmetic processes.

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[1] S. A. Courtis, *Standard Practice Exercises in Arithmetic* (Teachers' Manual), (Yonkers-on-Hudson, New York: World Book Company, 1916).

[2] W. L. Uhl, "The Use of Standardized Materials in Arithmetic for Diagnosing Pupils' Methods of Work," *Elementary School Journal*, Vol. 18, pp. 215–18.

[3] L. J. Brueckner, "A Necessary Step in the Diagnosis of Pupil Difficulties in Arithmetic," *Third Yearbook of the Department of Elementary School Principals* (Washington, D. C., 1924), pp. 290–300.

The steps to take in making a psychological diagnosis of the pupil's methods of work are as follows:

1. Select for special study the pupils in the class whose work is considerably below standard in any process or part of a process.

2. If the pupil is deficient in a whole process, select a standard diagnostic test containing a wide variety of types of examples in the process in which the pupil is deficient. When a standard test is missing, the teacher can devise an informal one for the immediate purpose. Several modern arithmetic textbooks contain satisfactory tests for this kind of diagnosis. If the pupil's deficiency seems to be limited to one element of a process or to a particular type of example, materials of a much more restricted nature, limited to the phase in which the pupil is deficient, should be used.

3. Select one of the pupils for special study. Bring him forward to the teacher's desk. Place him in a chair at the right of the examiner. At some convenient time, when the other pupils are working on drill exercises, direct the pupil to work the examples in the test aloud, simply saying to him, "Since we have discovered that you are not up to standard in ——, the best way for me to find how to help you will be for you to work the examples aloud for me, instead of working them silently, like this. . . ." After a simple demonstration, pupils readily understand what is wanted and proceed to work the examples.

4. Carefully observe the verbal statements of the pupil. Make a note of any faulty procedures, examples which evidently present serious difficulty, roundabout statements of method, or other possible sources of difficulty that are evident. Whenever necessary require the pupil

to repeat the work. If there is a long pause, the examiner should seek to determine what the mental activities of the pupil were during that time. This can usually be done by judicious questioning. To prevent fatigue, a diagnosis of only one process should be made on one day.

5. Observe the pupil's habits of work. He may be a "counter," he may dawdle, his attention may easily be diverted, he may be bored by the whole process. Throughout the entire period of the examination every effort must be made to take an impersonal attitude toward the pupil's responses. Under no circumstances should the pupil be censured for faulty habits that are revealed, since this might result in the inhibiting of his responses and make it difficult to proceed with the diagnosis. The attitude of the examiner should be the same as that of the physician or clinician who is seeking to determine the cause of an ailment, that is, a critical, impersonal study of the case in hand, unbiased by prejudices or emotion. Nor should any attempt be made during the diagnosis to teach the pupil the correct methods of work. This should not be done until after the diagnosis has been completed.

6. Record the results of the diagnosis on the pupil's record blank for future reference both in deciding what remedial work is needed and to find what faults have been eliminated as a result of the remedial work that is done. File the pupil's test paper with his record blank. (See pages 104 and 106 for typical record sheets.)

(b) *A typical summary of errors in a school.* The following statement is a summary[1] of the faults that were found as the result of the psychological diagnosis of the

[1] E. Probst, "Following Up a Survey of Instruction," *Proceedings of the Minnesota Society for the Study of Education* (Minneapolis, Minnesota, 1925), pp. 32–37.

difficulties of 45 pupils in grades 4 to 6 in the Calhoun School of Minneapolis who were rated by the test results as "pupils so far below standard that special adjustment is necessary":

The types of difficulty disclosed were most interesting. The most common fault proved to be the habit of counting. The teachers had worked faithfully to secure automatization of all combinations, but in spite of their efforts, twenty-three counters slipped through. They counted in the most amazing ways, with lips, tongue, toes, and fingers. Sometimes the counting was scarcely perceptible. Fourteen had a short attention span. They could readily add a column of four or five figures, but beyond that they were lost. Fourteen moved lips constantly, vocalizing every step, ten had a bad habit of guessing, eight failed because of faulty procedure, and six failed because of slowness.

Addition difficulties. Twenty-two skipped around, selecting combinations that seemed easy to them; eighteen hunted about for addends of 10; eighteen inspected the example to find a starting point; eleven had trouble with carrying; five added all the large numbers first to get them out of the way, and nine used curious roundabout methods.

Subtraction difficulties. Fifteen showed weakness in the fundamentals; fourteen had trouble with borrowing; twelve used roundabout methods; three always subtracted the smaller numbers from the larger whether it was in the minuend or the subtrahend; four added to obtain results, and three counted backwards, using the fingers to keep track of the count.

Multiplication difficulties. Ten showed weakness in fundamentals; seven had carrying difficulties; nine used the multiplicand and the multiplier, and two had zero difficulties.

Division difficulties. Nineteen had trouble with divisions having remainders; twelve had zero difficulties; eleven repeated the tables to secure results; eight used roundabout methods; twelve had difficulty with trial division, and six

couldn't remember the number to carry in the multiplication involved.

It took from forty-five to ninety minutes to complete an individual diagnosis, the time depending upon the number and kind of difficulties encountered. We were fortunate in having the assistance of student examiners, but if we had not had their help, we could readily have made our own diagnoses. Anyone who is supplied with the necessary diagnostic material can do the work. As a matter of fact, it is a distinct advantage for a teacher to make her own diagnosis, and some of our teachers preferred to do so.

The children, themselves, were keenly interested in the analysis and coöperated willingly with the teachers in their effort to improve the situation. It sometimes happened that a child had only one or two special difficulties. When these were known it was a comparatively easy matter to clear up the trouble. On the other hand, one boy in 5A grade had a total of twenty-three separate kinds of trouble. No wonder his teachers considered him extremely "careless" in the handling of figures!

(c) *Types of errors found.* Many peculiar errors, faulty types of procedure, and special difficulties are found in the work of deficient pupils. The following illustrations[1] will make clear some of the faults most commonly found:

DESCRIPTION OF DIFFICULTIES IN ADDITION

1. *Weakness in combinations.*

(a) Illustration:

$$\begin{array}{r} 9 \\ \underline{7} \\ 15 \end{array}$$

(b) Description.—Pupil does not know facts.

2. *Counting.*

(a) Description.—Pupil gets answer by tapping pencil, counting on fingers, moving lips, etc.

[1] Taken from L. J. Brueckner, *Diagnosis of Difficulties in Whole Numbers*, Manual (Minneapolis, Minnesota: Educational Test Bureau, 1925).

3. *Vocalization.*
   (a) Description.—Lip movement, audible or silent.
4. *Adds same digit to both columns.*
   (a) Illustration:

$$\begin{array}{r} 12 \\ 3 \\ \hline 45 \end{array}$$

   (b) Description.—Pupil adds, "3 plus 2 and 3 plus 1."
5. *Bridging.*
   (a) Illustration:

$$\begin{array}{r} 36 \\ 8 \\ \hline 53 \end{array}$$

   (b) Description.—Pupil says, "36 plus 8 are 53."
6. *Zero difficulty.*
   (a) Illustration:

$$\begin{array}{r} 40 \\ 78 \\ \hline 110 \end{array}$$

   (b) Description.—Pupil says, "8 plus 0 are 0."
7. *Breaks up combinations.*
   (a) Illustration:

$$\begin{array}{r} 66 \\ 47 \\ 99 \\ \hline 212 \end{array}$$

   (b) Description.—Pupil makes combinations that he knows. He says, "9 plus 3 are 12"; "12 plus 4 are 16"; "16 plus 6 are 22."
8. *Roundabout methods of work.*
   (a) Pupil has very indirect methods of work. This difficulty is closely allied to difficulties 7, 10 (1), 10 (4), and 10 (6).
9. *Carrying difficulties.*
   (1) Forgets to carry.
      (a) Illustration:

$$\begin{array}{r} 68 \\ 15 \\ \hline 73 \end{array}$$

(b) Description.—Pupil writes the 3 of 13, but does not carry the 1.

(2) Adds carried number irregularly.

(a) Description.—Sometimes adds carried number at the beginning of the next column and sometimes does not add it until he has completed adding the next column.

(3) Carried wrong number.

(a) Illustration:

```
 89
 36
---
161
```

(b) Description.—Pupil puts the 1 in the sum and carries the 5.

10. *Difficulty in Column Addition.*

(1) Adds large numbers first.

(a) Illustration:

```
 89
 43
 77
---
209
```

(b) Description.—Pupil says, "9 plus 7 plus 3," or "8 plus 7 plus 4." Evidently pupil does not know higher combinations.

(2) Trouble with second addition in column.

(a) Illustration:

```
 66
 47
 99
---
210
```

(b) Description.—Pupil says, "9 plus 7 are 16; 16 plus 6 are 20"; or "9 plus 2 are 11; 11 plus 4 are 14; 14 plus 6 are 21." Cannot add a written number to a thought-of number.

(3) Repetition.

(a) Description.—Pupil repeats addition of column because of short memory or attention span.

(4) Adds by tens.

(a) Illustration:

```
  114
    5
7,365
   59
-----
```

(b) Description.—Pupil says, "5 plus 5 are 10; 10 plus 9 plus 1 are 20; 20 plus 3 are 23."

(5) Loses place in column.

(a) Description.—Most likely to occur in long column addition. May be due to weakness in fundamentals, short memory span, break in attention, or lack of visual concentration.

(b) Remedial.

1. Begin with easy examples like:

```
2    3
1    4
3    2
2    1
1    4
2    3
-    -
```

(6) Inspects example to find starting point.

(a) Illustration:

```
 88
  2
877
354
---
```

(b) Description.—Pupil is confused as to the combination with which he is to begin. Shall he start with 4 plus 7, or 4 plus 2, or will 8 plus 2 be an easier combination to begin with?

Description of Difficulties in Subtraction

1. *Weakness in combinations.*

(a) Illustration:

```
8
3
-
6
```

(b) Description.—Failure to give accurate and quick response to the subtraction combinations. Pupil either guesses to get results or remains silent until given assistance by the teacher.

2. *Counting.*

(a) Illustration:

$$\begin{array}{r} 14 \\ \underline{7} \\ 7 \end{array}$$

(b) Description.

(1) Pupil deducts one from minuend recording a mark for each deduction until minuend is same number as the subtrahend, and then adds the marks to get the answer. The reverse of procedure may also occur when pupil adds to the subtrahend, recording marks until subtrahend is same number as minuend, and adding marks to get answer.

(2) Pupils also count with fingers, tongues, movements of the head, feet, etc.

3. *Zero difficulty.*

(a) Zero in minuend.

(1) Illustration:

$$\begin{array}{r} 100 \\ \underline{81} \\ 29 \end{array}$$

(2) Description.—Zero difficulties may be in units, tens, or hundreds place in minuends having two to five digits.

(b) Zero in subtrahend.

(1) Illustration:

$$\begin{array}{r} 8 \\ \underline{0} \\ 0 \end{array} \qquad \begin{array}{r} 24 \\ \underline{10} \\ 10 \end{array} \qquad \begin{array}{r} 324 \\ \underline{107} \\ 207 \end{array}$$

(2) Description.—In all three of the above samples pupil may say it is impossible to take nothing from something, hence will put down a zero for an answer.

4. *Borrowing difficulty.*

(a) Not allowing for borrowing.

(1) Illustration:

26	53
17	16
19	47

(2) Description.—Pupil fails to realize that we cannot take 7 from 6 but that we must borrow one from the two in the first sample. The same holds true in the second sample.

(b) Failure to borrow, giving zero in answer.

(1) Illustration:

32
16
20

(2) Description.—Pupil says he cannot take 6 from 2 so puts down zero as part of answer, and then subtracts the 1 from the 3 in the second digit, getting 20 for remainder.

(c) Deducting from minuend when borrowing is not necessary.

(1) Illustration:

38	746
5	213
23	423

(2) Description.—Pupil does not understand the "why" or the "when" of borrowing; has difficulty in forming new bonds in subtraction but when the new bond is learned, he has difficulty in going back to the old.

(d) Deducting two from minuend after borrowing.

(1) Illustration:

43	568
18	129
15	429

(2) Description.—Pupil does not understand the process of borrowing.

(e) Increasing minuend digit after borrowing.

(1) Illustration:

13	97
4	9
29	108

(2) Description.—Pupil does not understand the borrowing process.

(f) Errors due to minuend and subtrahend digits being the same.

(1) Illustration:

784	264
86	34
708	220

(2) Description.—Pupil does not know when to borrow or when not to borrow.

5. *Subtracting minuend from subtrahend.*

(a) Illustration:

62	145
7	82
65	143

(b) Description.—In these examples the pupil has taken the smaller from the larger number to get the results regardless of their position in the problem.

6. *Using same digit in two columns.*

(a) Illustration:

65	88
4	7
21	11

(b) Description.—Pupil subtracts subtrahend from each of the digits in the minuend to get results.

7. *Roundabout methods of work.* (Deriving unknown from known combinations.)

(a) Illustration:

$$\begin{array}{r} 10 \\ \underline{6} \\ 4 \end{array}$$

(b) Description.—Pupil says, "10 − 4 is 6; 6 − 4 is 2; and 6 − 2 is 4."

8. *Splitting up of numbers.*
   (a) Illustration:

$$\begin{array}{r} 347 \\ 94 \\ \hline 253 \end{array} \qquad \begin{array}{rr} 34 & 7 \\ 9 & 4 \\ \hline 25 & 3 \end{array}$$

   (b) Description.—Pupil evidently cannot subtract a two-digit from a three-digit number so splits up digits in minuend and subtrahend and subtracts from each division.

9. *Reversing digit in remainder.*
   (a) Illustration: 86 − 4 equal 28.
   (b) Description.—Pupil reverses numbers in units and tens place in remainder. This error would probably not occur in column subtraction.

10. *Confusing processes with division.*
   (a) Illustration:

$$\begin{array}{r} 86 \\ 4 \\ \hline 34 \end{array}$$

   (b) Description.—Pupil says, "4 goes into 16 four times," and borrows one from the 8, then subtracts the 4 from 7 getting 3, or a remainder of 34. The pupil has confused division with subtraction.

11. *Skipping one or more decades.*
   (a) Illustration:

$$\begin{array}{r} 15 \\ 13 \\ \hline 12 \end{array}$$

   (b) Description.—Pupil says, "13 from 15 is 12," thus getting an answer one decade too high.

Description of Difficulties in Multiplication

1. *Weakness in combinations.*
   (a) Illustration:

$$\begin{array}{r} 74 \\ 7 \\ \hline 508 \end{array}$$

   (b) Description.—Pupil says, "$7 \times 4 = 28$; $7 \times 7 = 48$ and 2 are 50."

2. *Counting.*
   (a) Carrying.
      (1) Illustration:
$$\begin{array}{r} 28 \\ 8 \\ \hline 224 \end{array}$$
      (2) Description.—Pupil says, "$8 \times 8 = 64$; $8 \times 2 =$ 16, 17, 18, 19, 20, 21, 22."
   (b) To get combinations.
      (1) Illustration:
$$\begin{array}{r} 63 \\ 6 \\ \hline 378 \end{array}$$
      (2) Description.—Pupil says, "$6 \times 3 = 18$; 6×5 $= 30$, 31, 32, 33, 34, 35, 36, and 1 to carry $= 37$."
3. *Zero difficulties.*
   (a) Zero in multiplier.
      (1) Illustration:
$$\begin{array}{r} 35 \\ 20 \\ \hline 35 \\ 70 \\ \hline \end{array}$$
      (2) Description.—Pupil says, "$0 \times 5 = 5$; $0 \times 3 =$ 3; $2 \times 5 = 10$; $2 \times 3 = 6$ and $1 = 7$."
   (b) Zero in multiplicand.
      (1) Illustration:
$$\begin{array}{r} 803 \\ 2 \\ \hline 1626 \end{array}$$
4. *Carrying difficulties.*
   (a) Carries wrong number.
      (1) Illustration:
$$\begin{array}{r} 632 \\ 9 \\ \hline 5958 \end{array}$$
      (2) Description.—Pupil carries wrong digit. He says, "$9 \times 2 = 18$; $9 \times 3 = 27$, and 8 are 35; $9 \times 6 = 54$, and 5 are 59."

(b) Forgets to carry.

(1) Illustration:

$$\begin{array}{r} 72 \\ 5 \\ \hline 350 \end{array}$$

(2) Description.—Pupil says, "$5 \times 2 = 10$; $5 \times 7 = 35$."

(c) Error in carrying into zero.

(1) Illustration:

$$\begin{array}{r} 807 \\ 7 \\ \hline 6009 \end{array}$$

(2) Description.—Pupil says, "$7 \times 7 = 49$; $7 \times 0 = 0$; $7 \times 8 = 56$, and 4 are 60."

5. *Errors in adding.*

(a) In partial products.

(1) Illustration:

$$\begin{array}{r} 13 \\ 12 \\ \hline 26 \\ 13 \\ \hline 166 \end{array}$$

(2) Description.—Pupil multiplies correctly but says, "2 and 3 are 6."

(b) In carried number.

(1) Illustration:

$$\begin{array}{r} 948 \\ 4 \\ \hline 3782 \end{array}$$

(2) Description.—Pupil says, "$4 \times 8 = 32$; $4 \times 4 = 16$, and 3 are 18; $4 \times 9 = 36$, and 1 are 37."

(c) Forgets to add partial products.

(1) Illustration:

$$\begin{array}{r} 21 \\ 32 \\ \hline 42 \\ 63 \\ \hline \end{array}$$

(2) Description.—Pupil multiplies correctly but fails to add partial products.

6. *Errors in multiplying.*

(a) Confuses products when multiplier has two or more digits.

(1) Illustration:

$$\begin{array}{r} 25 \\ \underline{26} \\ 530 \end{array}$$

(2) Description.—Pupil says, "$6 \times 5 = 30$; $2 \times 5 = 10$, and 3 are 13; $2 \times 2 = 4$, and 1 are 5."

(b) Splits multiplier.

(1) Illustration:

$$\begin{array}{r} 18 \\ \underline{60} \\ 1080 \end{array}$$

(2) Description.—Pupil says, "$0 \times 18 = 0$; $3 \times 8 = 24$; $24 + 24 = 48$.

(c) Uses multiplicand as multiplier.

(1) Illustration:

$$\begin{array}{r} 486 \\ \underline{3} \\ 3458 \end{array}$$

(2) Description.—Pupil says, "$6 \times 3 = 18$; $8 \times 3 = 24$, and 1 are 25; $8 \times 4 = 32$, and 2 are 34."

(d) Multiplies by adding.

(1) Illustration

$$\begin{array}{r} 42 \\ \underline{2} \\ 84 \end{array}$$

(2) Description.—Pupil says, "$40 + 40 = 80$, and 4 are 84."

(e) Multiplies by same digit twice.

(1) Illustration:

$$\begin{array}{l} 437 \\ \underline{230} \\ 13110 \\ 1311 \\ \underline{874} \end{array}$$

(2) Description.—Pupil says, "$0 \times 7 = 0$; $3 \times 7 = 21$; $3 \times 3 = 9$, and 2 are 11; $3 \times 4 = 12$, and 1 are 13; $3 \times 7 = 21$; $3 \times 3 = 9$, and 2 are 11; $3 \times 4 = 12$, and 1 are 13; $2 \times 7 = 14$; $2 \times 3 = 6$ and 1 are 7; $2 \times 4 = 8$."

7. *Omits digit in*

(a) Multiplier

(1) Illustration:

$$\begin{array}{r} 807 \\ 26 \\ \hline 4842 \end{array}$$

(2) Description.—Pupil fails to multiply by 2.

(b) Multiplicand

(1) Illustration:

$$\begin{array}{r} 618 \\ 159 \\ \hline 5562 \\ 618 \\ \hline \end{array}$$

(2) Description.—Pupil multiplies by 9 and 1, but neglects to multiply by middle digit 5.

(c) Product

(1) Illustration:

$$\begin{array}{r} 27 \\ 10 \\ \hline 27 \end{array}$$

(2) Description.—Pupil says, "$0 \times 7 = 0$ (not necessary to write it) $1 \times 7 = 7$; $1 \times 2 = 2$."

8. *Uses wrong process* (*adds.*)

(a) Illustration:

$$\begin{array}{r} 52 \\ 3 \\ \hline 85 \end{array}$$

(b) Description.—Pupil says, "$3 \times 2 = 5$; $3 \times 5 = 8$."

9. *Errors in position of partial products.*

(a) Illustration:

$$\begin{array}{r} 24 \\ 11 \\ \hline 24 \\ 24 \\ \hline 48 \end{array}$$

(b) Description.—Pupil places partial products under one another.

DESCRIPTION OF DIFFICULTIES IN DIVISION

1. *Weakness in combinations.*

   (a) Illustration: $7\overline{)56}$ with quotient 9

   (b) Description.—The pupil does not know the simple division facts.

2. *Difficulty with remainders.*

   (a) Within the example.

      (1) Illustration: $4\overline{)76}$ with quotient 11

      (2) Description.—The pupil does not carry the remainder of the first partial dividend to form the second partial dividend.

   (b) With final remainder.

      (1) Illustration: $2\overline{)587}$ with quotient 293

      (2) Description.—Pupil does not express remainder as a fraction or omits remainder.

3. *Zero difficulty.*

   (a) Within quotient.

      (1) Illustration: $6\overline{)642}$

      (2) Description.—Pupil omits zero in quotient. He says, "6 into 6, 1; 6 into 42, 7. Answer 17.

   (b) Within dividend.

      (1) Illustration: $3\overline{)60}$ with quotient 2; $30\overline{)60}$ with quotient 20; $2\overline{)804}$ with quotient 42

      (2) Description.—Pupil is confused by the zeros, giving 2 as the quotient of the first, 20 of the second, and 42 of the third.

4. *Difficulty with quotient.*

   (a) Trial quotient.

      (1) Illustration: $17\overline{)85}$ $23\overline{)437}$

(2) Description.—Pupil is unable to determine a quotient in long division when apparent quotient is not true quotient.

(b) Counts to get quotient.

(1) Illustration: $7\overline{)672}$

(2) Description.—Pupil does not know how many 7's in 67, so counts by 7's until he reaches 63, keeping count at the same time of the number of times he gave an exact multiple of 7, namely 9.

(3) Remedial.—The counting habit is evidence of weakness in the fundamental facts. To overcome firmly established counting habits, give oral division combination drill under time limit, decreasing the time with increasing skill of pupil.

(c) Derives quotient from a similar example.

(1) Illustration: As this may occur with any phase of division, sample is omitted.

(2) Description.—Not knowing the combination, the pupil looks for other examples already worked to find the answer in one of like combination.

5. *Roundabout methods of work.*

Under this caption may be designated any of the peculiar methods which individuals may show during diagnosis. On account of the variety of such methods and the fact that they are associated with individual characteristics, specific remedial measures cannot be prescribed here. Usually it may be sufficient to call the attention of the pupil to his particular peculiar method.

6. *Difficulty with subtraction.*

(a) Illustration:

$$\begin{array}{r} 40 \\ 23\overline{)1035} \\ 92 \\ \hline 15 \end{array}$$

(b) Description.—Pupil is unable to do subtraction to obtain partial dividend.

7. *Difficulty with multiplication.*

(a) Illustration:

$$\begin{array}{r} 56 \\ 32\overline{)1792} \\ \underline{160} \\ 192 \\ \underline{162} \\ 30 \end{array}$$

(b) Description.—Pupil is unable to do multiplication to prove trial quotient.

8. *Repeats tables for results.*

(a) Illustration:

$$\begin{array}{r} 7 \\ 6\overline{)42} \end{array}$$

(b) Description.—Pupil being unable to give the quotient repeated the tables of 6's to 42.

9. *Uses digits of divisor separately.*

(a) Illustration:

$$\begin{array}{r} 22 \\ 33\overline{)66} \end{array}$$

(b) Description.—Using digits in tens place, pupil says, "3 into 6, 2. Write down 2." Then he uses digits in units place and says, "3 into 6, 2. Write down 2." Answer becomes 22.

10. *Uses digits in dividend twice.*

(a) Illustration:

$$\begin{array}{r} 869 \\ 64\overline{)5568} \\ \underline{512} \\ 446 \\ \underline{384} \\ 628 \\ \underline{576} \\ 52 \end{array}$$

(b) Description.—Pupil becomes confused in long division and uses the 6 in the dividend twice.

11. *Interchanges long and short division.*

(a) Illustration: $6\overline{)749}$

(b) Description.—This is a short division example but pupil used the slower long division method.

(d) *Desirable types of supplementary information needed concerning the pupil.* Some pupils are so deficient in all phases of arithmetic that their mental level should first be determined by means of the excellent standard mental tests that are now available. Inferior mentality has been found to be a primary cause of poor work in arithmetic. It has also been found that even pupils of superior mentality sometimes show marked deficiencies in arithmetic. There are various reasons for this, such as gaps in their training due to frequent moving from town to town, indifference on their part to the formalized work in arithmetic, or excessive absence.

The physical condition of these pupils should also be studied, to determine whether their vision and hearing are normal, whether they are anemic or underweight, or whether special medical attention is needed. Their school history should be known, whether they have skipped grades, what grades they have repeated, the subjects in which they have done satisfactory or unsatisfactory work, the number of schools they have attended, and their general attendance. Their ratings in achievement tests in other subjects should also be known; for example, weakness in problem solving may be associated directly with inferior reading ability, in which case the latter is undoubtedly a *contributing* factor to deficiency in the former.

The pupils' general methods of work should also be observed. Their attention may be very poor, any simple distraction diverting them from the work at hand. They

may merely dawdle over their work, making no real effort to complete the assignments. Day dreaming, misconduct, whispering, and other forms of undesirable behavior can be detected. They may be nervous or work in a tense manner. Such symptoms show that the pupil has no effective habits of work and that a definite effort must be made to develop them before much improvement in arithmetic can be expected.

4. CASE STUDIES.

The following *case* studies adapted from those by Souba[1] and Martin[2] contain descriptions of the results of diagnostic studies of pupils who are markedly deficient in operations in whole numbers and in problem solving. A careful study of these cases will show the great variety of factors that may have a direct bearing on the quality of the pupil's work. Record blanks used for recording faults in addition of whole numbers and in addition of fractions are given on pages 104 and 106.

CASE A (after Souba)

A Fourth Grade Child with a High Intelligence Who Will Not Work Because He is Bored with the Drill Side of School Life

E. S. was nine years old in June, 1923, at the time of the examination. He came from a very wealthy home where the children are allowed to do as they please and where they have nurses and maids to wait on them. The parents are both highly educated and both are vitally interested in their careers. They are very indulgent in their attitude towards their children. E. S. has everything he desires at home.

[1] A. Souba, "Diagnosis of Difficulties in Arithmetic," Master's Thesis, unpublished (Minneapolis, Minnesota: University of Minnesota, 1923).
[2] C. Martin, "An Analysis of the Difficulties in the Arithmetical Reasoning of Fourth, Fifth, and Sixth Grade Pupils," Master's Thesis, unpublished (Minneapolis, Minnesota: University of Minnesota, 1927).

He has a chemical set at home in which at present he is greatly interested. He is always talking about the things he has made with this set. Just recently he made some ink. His greatest difficulty in school arises from the fact that he has never been made to conform to any rules at home. He does not know what it is to get down to work or to do anything that he does not wish to do at that particular time.

His attendance at school has never been regular. He attended a private school at first and then at the request of his parents attended only half-day sessions in a public school. He repeated the 2C and 3C grades because he was absent so much of the time those two terms, while traveling with his parents. At the time of the examination he was doing the work of two grades, the 3A and the 4C.

He does well in the subjects that are of interest to him. The novel and unusual situations appeal to him, but he tires soon even of those. He was interested in the mental test as long as it did not involve arithmetic, but his responses were so irregular that so many tests had to be given to find his mental level that he became tired and wished to quit before the examination was completed. He lacks in staying powers. He soon tires of everything. It was necessary to go back to the tests of the fifth year to find his basal year and he passed two tests in the fifteenth year. He has a mental age of ten and five-eighth years and an intelligence quotient of one hundred and eighteen. At that, it is reasonably certain that his score would have been higher if it had been possible to divide the examination period into several sessions so that his interest might have been sustained.

His teachers have the same difficulty with him. It is very difficult to keep him interested in the things going on in school. He does very well in the things that appeal to him. His teachers rate him A in intelligence, B in scholarship, and C in industry. He understands the processes in arithmetic, but the whole subject bores him. He only does the work under protest and his work is never completed on time. He has been on the first lesson in the Studebaker Practice Tests for sixteen days. A study of his record reveals some interesting points.

RECORD

Exercise	Date	Tried	Right
I	May 1	50	48
I	2	43	43
I	3	60	48
I	4	56	56
I	10	44	43
I	11	40	38
I	14	41	40
I	15	52	52
I	16	45	44
I	17	60	60
I	28	26	26
I	29	27	26
I	31	15	14
I	June 1	8	8
I	2	63	63
I	6	91	90
I	7	76	75
I	8	50	50
I	12	45	45

For one thing this record shows his irregular attendance. It also shows that he is able to do well if he so desires, but that his interest is not in the work or he would not allow his scores to go down to almost nothing. It is possible to pick out the days when he has a determination to do the work. Another noticeable fact is that he is apt to be quite accurate. He usually has practically all the examples he attempts correct. Anyone who can work eight examples one day and ninety-one another has possibilities that need to be directed. It is poor pedagogy to keep such a child on such a test for sixteen days.

He always has been a problem to his teachers. One day he was scolded by the principal for some minor offense and as a result he stayed home. He claimed he had a headache.

He has had glandular trouble, his tonsils have been removed and he has been fitted with glasses, but he does not wear them. He is a fine looking boy, large for his age. In so many ways he seems mature, while in other ways he is

nothing but a baby. His general attitude toward school work is indifferent. He does well in all phases of school work involving reading, but poorly in the drill phase of school activities, not because he is incapable of doing the work, but because it is of little interest to him.

In the time allowed for his grade test he attempted seventeen examples and had four correct in the Courtis Supervisory Test. He skipped the last four subtraction examples and worked the first three multiplication examples in the time allotted for his grade test. It took him eight minutes and thirty seconds to complete the twenty-nine examples. Only five of the entire set were correct. In the Minneapolis 3C and 4C Arithmetic Test he made a score of twenty-four. This places him at the 3B standard. His responses to the diagnostic tests were very slow, but his accuracy was good considering his lack of concentration. His attention was always wandering, he was continually fussing, wiggling and turning around. He kept singing out his results and repeating them over and over to himself, making a little song out of every example. He counted with his head and fingers and kept his place with his pencil. It was very evident throughout the entire examination that he was a child who had the ability, but who would not work.

In addition he broke up his numbers, added the small numbers first, grouped his numbers, had trouble with zeros and used roundabout methods. In subtraction he usually took the smaller number from the larger; he had trouble in borrowing; he used the Austrian method to some extent and counted up from the subtrahend. In multiplication he always called the upper number the multiplier. He seemed to have formed wrong associations with some numbers. He gave the results better if the examiner pointed to the examples and kept encouraging him all the time, in fact that seemed to be the only way to get him to work at any time. He wanted attention all the time. He had formed some wrong associations in division, but his main difficulty in division lay in dividing a number by itself.

E. S. is a child of keen intelligence, who unfortunately for his own good has always been a law unto himself. With

him it is not a question of inability to do the work. It is simply a matter of doing what is required even if it is a matter of little value to him. I believe E. S. would probably be able to gain all the arithmetic that was required in his particular grade in a few lessons or at most in a few weeks and that he would learn it if he knew he would be excused from the drudgery of the drill periods as long as he kept up to the standards required for his grade.

Case B (after Souba)

A Sixth Grade Boy With a Keen Mind Whose Mental and Physical Reactions Function Very Sluggishly

S. W. was eleven years and six months old when he was included in this study. His parents, who are Norwegians, were in very comfortable circumstances and lived in a good residential neighborhood. They were interested in the boy's progress in school because they were ambitious for him to attend college and to succeed in life. The father, especially, seemed a very intelligent man who greatly regretted his failure to obtain a college education while he was young. S. W. is an only child and everything is bought for him and done for him. He is a very well brought up child. He seems to be unspoiled in spite of the unusual care that is lavished upon him.

He is one of those children who seems incapable of quick action. He is very slow and deliberate in everything. If given time enough, he is able to do any task that is set for him, but he has always been a cause of exasperation to his teachers because he is so dreadfully slow. He has always made his grade. At the time of the examination he had a very sympathetic teacher who seemed to see great possibilities in him.

His general attitude is very good. He is phlegmatic, careful to the extreme, and dreamy. His teachers were apt to consider him lazy. He is always willing to coöperate, and while he does ask for help when he needs it, he is apt to be confident in his ability to perform his tasks and is apt to depend upon himself and not on others.

He is considered a "C" child in scholarship, intelligence, and industry by his teachers. He does very fine work in drawing. He does slow written work because he draws his letters and figures. He is a very quiet child. He says very little and often is not given credit for what he does know because he gives his answers in such a few words. His mother has always helped him at home with his work, especially in arithmetic.

The writer worked with S. W. six months before he was included in this study and diagnosed his difficulties in arithmetic for his teacher. When more cases with a higher level of intelligence were needed, the writer asked for S. W. The principal and teachers all said that he was not the type of case desired, that his intelligence was not above average, but they raised no objections to the administration of a mental test. S. W. fell into the highest level of intelligence while the three other cases picked by the principal and the teachers fell into the dull level of intelligence. This fact was a source of great pleasure to the parents, who were conscious of the boy's slowness and had often wondered if it was a matter of intelligence. His mental age was thirteen and one-eighth years and his intelligence quotient was one hundred fourteen. The writer felt sure that the scores would have been even better if the time element had not played such a large factor in the mental test.

S. W. attempted ten examples and had five correct when he first tried the Courtis Supervisory Test. The next time he tried the test he raised his score to twenty attempted, and eighteen right. When he was given all the time he wanted on this same test he had twenty-one examples correct. His accuracy is not very good.

In the diagnosis it was noticed that vocalization and lip movements were present to a slight degree. It was very evident that S. W. counted. In the simple diagnostic tests he did very poorly both as to speed and as to accuracy. This was especially noticeable in addition. He had many roundabout methods of doing his work. He inspected his column for the starting place, and would start adding one way, and then discover a difficult combination and stop and

begin adding from the other end. He skipped around the column, grouped his numbers, and then added the groups. He was weak in fundamentals and had formed the wrong associations for certain combinations. He had trouble with the second addition. He could not remember his sums and he had difficulty in carrying and in bridging tens. In all his work he made a long story instead of simply stating his results.

In subtraction he used faulty statements when he said he took the upper numbers from the lower. He very seldom did take the upper from the lower numbers, but he always stated his process in this way. He was weak in the fundamental combinations and evidently formed wrong associations. He had difficulty in examples involving borrowing. As a rule his results were ten too large. Examples involving the use of the zero were also a source of difficulty.

In multiplication he was apt to confuse this process with addition. In several of the examples he only used partial multiplication and finished the example with some other process. Carrying caused him trouble. Often he obtained the correct results, but was unable to remember his product and so laboriously had to repeat the whole process. He did not always use the multiplier as the multiplier, but inspected his numbers first to find the one that was the most familiar to him. He did not know his combinations, often failing on very simple ones as $3 \times 6 = 16$, or $3 \times 4 = 10$.

In division a large part of his difficulties were caused by his inability to subtract, expecially in examples involving borrowing. He did not know his combinations and had formed wrong associations. It seemed difficult for him to retain in mind his results. Examples involving the use of zeros also caused him trouble. In several examples he did not carry. Short division seemed to be more difficult than long division.

S. W. has a fine mind in a large, slowly moving body. He is dreamy and slow to respond and his slowness is a source of trial to his teachers. He is one of those children whose nerves and muscles will probably always function slower than normal. Little can be done for such a case except that

the standard should be set a little higher than the present achievement, a little at a time and a sympathetic encouragement should be given all the time to aid the child to better his own record. The class instruction has been too rapid for him to grasp all the points. The rest of the class has probably been ready for the work before he grasped what was wanted. His motor and mental reaction after he did get started, were so slow that he could not keep up with the class.

Case C (after Souba)

An Over-Age Boy Who Lacks a Foundation in the Fundamentals of Arithmetic

J. B. was sixteen years and six months old when a study was made of his case. He was normal physically and had attended school very regularly for ten years. He was a constant repeater from the fourth grade up. He repeated the 1B, 4B, 5C, 5B, 5A, 6C, and 7A grades.

According to the report of his teachers his comprehension in reading was above the average, but he seemed to lack a foundation in the mechanics of arithmetic. This weakness was always noticed and was the cause of much of his repetition of grades. His arithmetic has always seemed a confusion of ideas and while he has improved, as a result of individual instruction which has been given him, he has not by any means overcome his weakness in the mechanics of arithmetic.

His teachers rated him "B" in intelligence and in industry. According to the findings from the Kuhlmann Test he is a dull normal child. His general attitude is good and he is very unassuming in his manners. He seems to have quite a talent in drawing and wishes to study further along that line, but he lives with his mother and grandmother and they are not able to pay for his further schooling. He will have to work, but he does not seem to know how to obtain a job. His grandmother seemed very much concerned about his future. She feels that J. B. will not be able to keep his position, if he obtains one, because he is unable to tell time. She wondered why the schools did not teach time. The difficulty with J. B. seems to lie largely in the fact that he

realizes that he has missed learning things that others learn much younger. As he is sensitive about this and is ashamed to ask for help now or to allow others to help him, it will be difficult for him to learn these things at the present time. A sympathetic stranger could do more with him now than his own people. They evidently have scolded and ridiculed his shortcomings.

J. B.'s scores in the Courtis Supervisory Test in two trials were twenty-one attempted and thirteen right, and nineteen attempted and ten right, respectively. When the same test was repeated in the diagnosis, he tried seventeen in the time allotted for his grade, and had twelve correct. It required six and one-fourth minutes to complete the test, the same time that is given to low fourth grade children, and then he had only fifteen examples correct.

Vocalization and lip movement as well as counting were very noticeable in the examination of his difficulties in arithmetic. His speed and accuracy were both very low. Guessing and a lack of thoroughness in all the four operations were common characteristics of his work. He used roundabout methods, skipped around in the column, broke up his numbers, combined numbers into groups, and added the groups, broke up numbers and recombined these parts with other numbers, counted up by ones, and tried other numbers first. He had difficulty in carrying and in second addition. His attention span was short. He was weak in his combinations and had formed wrong associations.

In oral subtraction most of his responses were incorrect, probably due to the fact that he did not take time to count out his answers. He did not know his combinations and seemed uncertain as to the method. At times he used the Austrian method of subtraction. Most of his difficulties were due to confused ideas in examples involving the use of zeros and borrowing.

In multiplication the multiplier was not always used as the multiplier. At times he became confused and used only partial multiplication. He did not know his multiplication combinations and had to repeat his tables in order to obtain those that he did know. He was absolutely unable to do

long division. He did have some success with short division, but it was a very laborious process. He repeated the tables for the results, had difficulty with uneven division, trial division, and with zeros, did not carry, forgot what to carry, and paid no attention to the remainders.

J. B. is an over-age child whose arithmetical sense has been slow in developing. Possibly little individual instruction fitted to his particular needs was given to him when it would have done the most good. The gains from even a large amount of drill at the present time are apt to be small.

Case D (after Martin)

An Over-age Child Deficient in All Phases of Arithmetic

D. D. was selected for this study because of the unusual and rather inexplicable results on his test papers in arithmetic reasoning. D. D. is the oldest child in a family in which there are five children. The family lives in a rented house in a rather undesirable neighborhood, made so by the proximity of factories, railroad trackage, and tumble-down, dilapidated houses. The father is a practical engineer and works in a factory not far removed from the home. He and the mother, too, are very much interested in the welfare of the children and coöperated with the writer in every possible way to see if something might come of the interviews that would be of value to the child. The father seemed to have lost patience with D. D. because he could not grasp the fundamentals of arithmetic and this attitude hindered rather than improved the boy's chances for development. The mother was conscientiously interested in her children and helped them as much as she could with their school work when they brought it home. This was especially true in the case of D. D. The mother stated that he very seldom had difficulty with his other subjects, but that arithmetic seemed to be his "bugbear" and that he could not seem to understand it. A blackboard had been provided for his use at home and it was here, the mother stated, that she helped him in the evenings to attempt to master the different combinations in the four fundamentals. D. D. could not seem

to retain the combinations for any length of time. The mother helped him to memorize them in the evening until he was tired and she was sure that he had learned some of them, but the next day he would have forgotten them. D. D. is a boy who becomes very easily discouraged. He does not seem to have much real "backbone" and when he could not get the combinations quickly, he seemed to lose interest and to become sulky.

D. D. is a boy of average intelligence, his I.Q. in the Haggerty Delta 2 being 99. His low achievement in arithmetic may be partially due to the fact that he has attended five different schools since he began. This perhaps has only a very indirect bearing upon the case, however, because his achievement in his other subjects seems to be on a par with his intelligence. D. D.'s progress in school has been entirely normal, i. e., he has not had any repeats or skips. He has secured average marks in his other school subjects during his attendance but has been conditioned twice on account of poor work in arithmetic. He is very much interested in Elementary History and enjoys reading stories of historic battles and heroism.

D. D. is not an especially robust child. His tonsils and adenoids have been removed and his eyes have been corrected with glasses, but he does not wear the glasses because they make him nervous. He has had kidney trouble, his mother states, and this, together with his nervousness, has aided in hindering his development. However, he enjoys getting outside with the other boys. He is very fond of baseball and spends all of his spare time, when the weather is favorable, on the corner lot engaging in this pastime.

D. D. scores 0 in the Buckingham Scale for Arithmetic Problems. His score in the Stanford Reasoning Test was 8, the standard in this test being 36 for his grade. In other words, he worked two reasoning problems correctly when he should have been able to work nine to comply with the standard. He was above normal in the two reading tests, his score in the Stanford Reading Test being 95 with a standard of 84, and his scores for rate and comprehension in the Monroe Silent Reading Test being 125 and 8, respec-

tively, the corresponding standards being 122 and 7.7. One of D. D.'s serious weaknesses in arithmetic lies in his inability to cope with the four fundamental processes. His scores in the addition, the subtraction, the multiplication and the division sections of the Woody-McCall Scale were 6, 0, 3, and 2, respectively, the standards in those tests for his grade being 17.8, 15.7, 11, and 10.5. His weakness in fundamentals was given further emphasis by his score of 20 in the fundamentals test of the Stanford Achievement Test, the standard for that test being 65. In the Stevenson Problem Analysis Test his score and the standard for his grade were 7 and 11, respectively. His condition in arithmetic is apparently pitiful.

An inspection of the fundamentals test of the Stanford Examination indicates that D. D. did not have a weakness in any particular process, but weaknesses in all of them. For instance, he secured 5 when he added 3 and 2, but 9 when he applied the same process to 3 and 4. When he subtracted 2 from 4, his result was 5; 4 from 7 equaled 8; 5 from 16 equaled 14, according to his figures. His work in the other processes closely corresponds to these results. Sometimes he was fortunate enough to get the right result; more often he was not. He attempted every one of the forty-seven problems in this test and only five of them were correct. A fourth-grade pupil could not possibly hope to work some of the more advanced problems in this test, and yet D. D. blindly attempted all of them; that is, he wrote an answer under each one. He seemed to think that he was required to work all of them and he did not want to cause displeasure.

D. D.'s case seems just as hopeless in reasoning. It naturally would seem hopeless when he does not have better command of the fundamentals than he does. The test was not a reasoning test to him but a guessing contest. D. D. answered the first two questions of the Stanford Reasoning Test correctly. He did know these combinations correctly for the writer quizzed him regarding them some days after the test.

The third problem reads, "A hen had 9 chicks and 3 of

them died. How many were left?" His answer to this problem was "9," a mere guess.

In the fourth problem he divided when he should have added. His division, however, was correct.

In the fifth problem which reads, "If you buy a pencil for 4 cents and pay for it with a dime, how much change should you get?" D. D.'s result was 9.

In this problem there were three things involved: the knowledge of the value of a dime, the selection of the correct process (subtraction), and the carrying out of that process successfully. D. D. knew the value of a dime to be ten cents, and when asked, after the test, the proper process to use, he suggested subtraction, which was right. When asked to solve it, he gave a result of 6, his answer the first time being 9. This indicated, very conclusively, that he was guessing.

The probabilities are that he guessed at the process also for he did not seem to know the correct process in other problems. In the test he stated that there are 3 dimes in a dollar, but when asked afterwards he knew the correct response.

In the problem, "How many eggs are there in 7 nests if each nest has 3 eggs?" his response in the test was 2. He did not know what process to use when he later was asked about the problem. When asked to work it, his figures looked something like this: $3 + 3 = 6 + 3 = 8 + 3 = 11 + 3 = 14 + 3 = 17 + 3 = 20$.

He seemed to know how to get the right result and would have secured it if he had not made a mistake in adding 6 and 3. He did not know that he was supposed to use multiplication, however, and when told, he did not know the product of 7 and 3.

In the next problem which involved the product of 8 and 3, his result was 9. When asked to rework the problem, he gave 28 as the result. After the problem was explained to him, he was asked the product of 8 and 3 and his response was 23.

D. D. gave answers to all of the forty problems in the reasoning test. His results were just as erratic as in the

problems above; more so, if possible. His results in the Stevenson Problem Analysis Test would indicate that in most cases his silent reading ability enables him to find out what is given in the poblem and what he is to find, but he cannot place his finger upon the proper process for solution, nor can he execute the processes correctly. D. D. seems to be the product of poor teaching in arithmetic, or perhaps he lacks capacity to think in mathematical terms.

Case E (after Martin)

A Superior Child with a Dislike for Arithmetic

H. P. is a precocious child of grade 5B who has a genuine dislike for arithmetic because he has not been able to master it. He is a very likable boy with rosy cheeks and flashing brown eyes. The family, consisting of a father who does not live at home, the mother and two children, a boy and a girl, is in rather unfortunate circumstances. H. P.'s father is an expert candymaker and has an excellent income. The separation between the father and mother was discussed very freely by the members of the family with no apparent feeling of regret or disturbance.

H. P.'s parents are both Italians, having emigrated from Italy shortly before H. P. was born. The father has good command of the English language considering the handicap of coming into a new country at an advanced age, but the mother has not been so fortunate. Neither the father nor the mother had a great deal of schooling, and while the mother seems to want H. P. to go on to school, she is rather negligent of some of the things necessary to make a child's education a success. For instance, both children had been attending summer school for about three weeks when the writer first visited the home, and during that time the children had been tardy a large portion of the time. The mother stated that she forgot to get them up at the proper time, and the children did not get up of their own accord because they were permitted to stay up until eleven o'clock or midnight every evening. What child will arise early every morning to attend summer school if his mother

permits him to retire at a late hour and does not make a serious effort to get him started for school at the proper time?

H. P. has lived in Minneapolis for four years, having come here from New York City. During his residence here he has attended three different schools. His school progress has been normal with one exception—he was permitted to skip grade 2A. One would think H. P. to be an exceptional scholar because his I.Q. is 130, the highest in the group of fifty-six pupils. This is not the case, however, his school work being only of mediocre caliber. H. P. received C's in all of his work the last school year with the exception of arithmetic and spelling, in which he received F and A, respectively. H. P. is permitted to do pretty much as he likes as far as home study is concerned. He does not like arithmetic and language so he never brings these texts home; however, he brings home the books of those subjects in which he is interested. He does a great deal of reading at home, securing his books at one of the Minneapolis Public Libraries. H. P. has ambitions as far as his future is concerned and might be able to fulfill them were he to get the proper encouragement and backing at home. He would like to be either a lawyer or a musician some day, he stated. These boyish aspirations are not always significant, but the fact that they are present is a very good indication. H. P. is such a sincere, courteous little chap that he makes one wish that his aspirations would eventuate in something worth while.

H. P. seems to be in good physical condition. He has had many of the diseases which children usually have but they have left no ill effects. Baseball, football, and wagon-coasting are his favorite outdoor sports and he spends a great deal of time engaging in these.

H. P. is far above the average for his grade in reading, his scores in the rate and comprehension sections of the Monroe Silent Reading Test being 188 and 15, respectively, as compared with standards of 142 and 9.8. He did equally well in the Stanford Reading Test, his score and the standard being 153 and 119, respectively.

H. P. was above the standard in the Stanford Computation Test, but his scores of 18, 19, 15, and 9 in the four divisions of the Woody-McCall Scale were below the corresponding standards of 21.4, 19.4, 17.2, and 16.5 for his grade. His weakest process was evidently division. He was slightly above the standard in the Stanford Reasoning Test but scored 0 in the Buckingham Test and 10 in the Stevenson Problem Test, the standard in this test being 16. He missed several problems in these tests because of confusion of processes and inability in the fundamentals, and others because he arranged his work so carelessly.

In the first problem of the Buckingham Test which involved the division of 36 by 12, he subtracted the 12 from the 36, an evident choice of incorrect process.

The next problem of the same test read, "An automobile was run 30 miles every day for a week. How many miles did it go?" H. P. again chose the wrong process, adding 30 and 7 instead of multiplying.

The next problem was a two-step problem involving the multiplication of 8 by 5 and the division of this result by 4. Here, he added the 8 and 5, indicating that he did not understand the problem. When asked why he had done this he stated that he did not know. The writer explained the problem to him and gave him another problem of similar type which he succeeded in working correctly.

The next problem read, "Ned sold his rabbit for 30 cents. This was $3/5$ of what he paid. How much did he pay for the rabbit?" After the test H. P. said that he did not understand the use of fractions. He felt that he ought to work the problem, however, so he added 30 and 35 because 35 and $3/5$ looked practically the same to him. His result for this problem was 65.

In the next problem, which involved the addition of four items involving dollars and cents, H. P. would have been able to work the problem if he had arranged his work carefully, for he did so afterwards. Many of H. P.'s problems were difficult to interpret as he had figures strewn all over the page. In this case, he was asked to work the problem over while the writer looked on. In another problem of

the Buckingham Test which involved the finding of $\frac{1}{3}$ of 210, H. P. again stated that he did not understand fractions. In another problem involving the subtraction of $18\frac{1}{4}$ from 42, H. P. chose the correct process but was again troubled with fractions. He gave a result of 24, neglecting the $\frac{1}{4}$ altogether.

It seems rather peculiar that a 5B grade pupil of his intelligence could not understand simple fractions like $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

H. P.'s teacher volunteered the information that he had a rather bored attitude during the arithmetic classes and that she could not seem to interest him or challenge his thinking powers.

H. P. attempted four problems in the Stevenson Problem Test. He knew what facts were given in two of these; knew what to find in all four; knew the most reasonable answer in two, and chose the correct process for solution in two. All that is involved is the ability to comprehend and H. P. has that ability. Had he been interested in the problems, and concentrated his attention on them, he should have been able to tell what facts were given in all the problems he attempted.

H. P.'s principal difficulties in arithmetic reasoning are carelessness, inability in fractions, and an evident lack of interest in all that deals with arithmetic. Were one able to instil in him an interest for the subject, his other difficulties could be removed speedily.

BRUECKNER DIAGNOSTIC TEST IN ARITHMETIC

Record of Individual Diagnosis—Whole Numbers

Addition Difficulties

Name.................... Grade............ Room............ Date....................

Ages: Chronological.................... Mental.................. I. Q...........

General Rating by Teacher: Scholarship................ Effort.............

Intelligence..

Set	Score	Diagnosis	Classification of Difficulties	Remarks
1			1. Weakness in combinations.	
2			2. Counting.	
3			3. Vocalizes his work.	
4			4. Bridging the tens.	
5			5. Zero difficulty.	
6			6. Breaks up combinations.	
7			7. Roundabout methods.	
8			8. Carrying difficulty:	
9			(1) Forgets to carry.	
10			(2) Adds carried number irregularly.	
11			(3) Carries wrong number.	
12			9. Column addition:	
13			(1) Adds large numbers first.	
14			(2) Trouble with second addition in column.	
15			(3) Forgets sum and repeats work.	
16			(4) Adds by tens. (5) Loses place in column.	
17			(6) Inspects example to find starting point.	
18				

Difficulties: a. Major b. Minor	Directions: Under "Score" indicate the number of examples wrong in each set. Under "Diagnosis" indicate by number the types of difficulties found in each set.

General Remarks:

BRUECKNER DIAGNOSTIC TEST IN FRACTIONS

Individual Diagnostic Record Sheet—Addition

Name, Lowell H——— Grade 8A Room 304

School, Franklin Date, March 8, 1928

Diagnosis	Summary
I. Lack of Comprehension of Process	
a. Adds numerators and denominators	..
b. Adds numerators, multiplies denominators	..
c. Numerator added without changing to common denominator. Either denominator used in sum	25
II. Reduction to Lowest Terms	
a. Fraction not reduced	3
b. Denominator divided by numerator	..
c. Denominator and numerator divided by different numbers	..
III. Difficulties with Improper Fractions	..
a. Not changed to mixed numbers	..
b. Changed but not added to whole number	1
IV. Computation Errors	
a. Addition	9
b. Subtraction	..
c. Division	..
V. Omitted	..
VI. Wrong Operation	..
VII. Partial Operation	
a. In adding mixed numbers adds only fractions	..
VIII. Changing to Common Denominator	..
IX. Other Difficulties	..

First indicate by number opposite each row the types of errors made on each example that was missed. For example, Ia means that the pupil adds numerators and denominators.

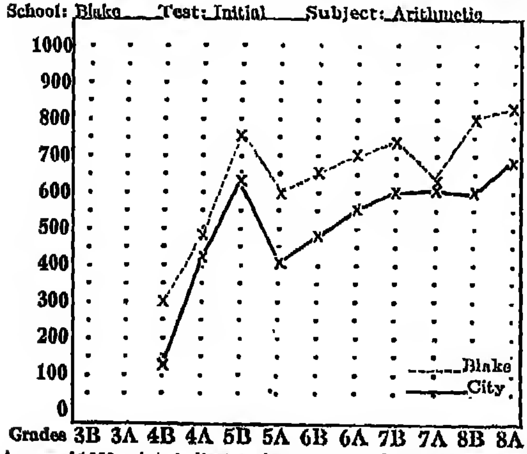
Then summarize under "Summary" the total number of times each difficulty was found.

EXAMPLES

Row	1	2	3	4	5
I			2a		2a
II	4a				2a
III			4a	3b	.
IV	1c	1c	1c, 4a	1c	1c
V	1c	1c	1c, 4a	1c	1c
VI	1c, 4a	1c	1c, 4a	1c	1c, 4a
VII	1c	1c, 4a	1c	1c	1c
VIII	1c	1c	1c	1c, 4a	1c
IX					

DETROIT PUBLIC SCHOOLS—Dept. of Educational Research
SCHOOL REPORT AND GRAPH SHEET: SCORES.

School: Blake Test: Initial Subject: Arithmetic



A score of 1000 points indicates that every child in the class has reached the standard level of ability for his grade.

POINT SCORES

Scores Test

Grades	Room	School	City	Differences: Above City	Differences: Below City
3B	1				
3A	2				
4B	3	300	135	105	
4A	4	475	427	48	
5B	5	750	631	119	
5A	6	600	406	104	
6B	7	650	480	170	
6A	8	700	558	142	
7B	9	720	605	115	
7A	10	640	618	22	
8B	11	800	600	200	
8A	12	825	670	155	

Problems for Study, Reports, and Discussion

1. What is the difference between diagnosis and measurement?

2. Describe a technique for selecting pupils for diagnosis.

3. What facts should the teacher secure to aid in the making of a diagnosis?

4. Why is a test score an inadequate method of making a diagnosis?

5. Why are survey tests not satisfactory diagnostic exercises?

6. Describe several useful diagnostic procedures and point out the limitations and value of each.

7. Secure a set of test papers and attempt to make an analysis of the causes of errors in the examples solved incorrectly. What difficulties do you encounter in making the analysis? Are you sure that your diagnosis is correct?

8. How do you account for the roundabout methods of work that have been discovered in pupil's procedures in solving examples?

9. Suggest some things that a teacher may do to prevent the development of faulty methods of work.

10. Show that difficulty in long division may be due to weakness in subtraction.

11. How do tests help the teacher to instruct more effectively?

12. If a class score is considerably above the average, is it a sign of good teaching?

13. If a class score at the end of the year is below the average, is it a sign of poor teaching?

14. Why do the abilities of pupils in a class vary so greatly?

15. Give a standard test and analyze the results. Prepare a chart to aid the pupils to interpret the scores.

16. Give arguments for a plan of completely individualizing instruction.

17. How does the teaching of arithmetic in the Winnetka schools differ from the conventional teaching of the subject?

18. What should a teacher do for pupils whose test scores are considerably above the standard?

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CHAPTER IV

THE DIAGNOSIS OF DIFFICULTIES IN THE PROCESSES WITH WHOLE NUMBERS

In order to make an adequate diagnosis of the causes of failure or deficiency in the four fundamental operations with integers, the examiner should have: 1. A clear understanding of the specific skills and abilities which the process under consideration includes; 2. A knowledge of the most common faults and difficulties that have been found in that process, and their symptoms; and 3. An adequate, reliable diagnostic test together with the necessary blanks for recording the faults that are noted in the pupil's work. No diagnosis should be considered complete until the necessary remedial work has been prescribed.

A. ADDITION OF WHOLE NUMBERS

1. ANALYSIS OF THE ADDITION PROCESS.

An excellent classification of the skills in each of the four processes with whole numbers, not including long division, was published by Miss Elda Merton[1] in the *Second Yearbook of the Elementary School Principals*, 1923. The specific abilities listed for addition are as follows:

[1] E. Merton, "Remedial Work in Arithmetic," *Second Yearbook of the Department of Elementary School Principals* (Washington, D. C., 1923), pp. 395–411.

1. The 100 addition combinations.
2. Ability to apply the combinations to higher decades:

2	52
4	4

3. The meaning of the addition sign.

4. The meaning of the following terms: *Addition, sum, add, addend,* and *carrying*.

5. That in writing the example, units must be placed under units, tens under tens, etc.

6. That one must begin at the right and work to the left.

7. That unit figures should be added to unit figures in a column, tens to tens, etc. This also includes the ability to keep one's place in a column.

8. Ability to add a seen to a thought-of number: After a child has added 3 and 6, he no longer sees both of the numbers he is required to add. Now 7 is a seen number, but 9 is only a thought-of number. This also includes the ability to keep in mind the result of each addition until the next number is added to it.

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47
56
93
..
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9. How to regard zeros in a column.

10. How to regard empty spaces in a column.

11. How to place the answer in the sum when a column has been added and the total sum of these figures is less than 10.

12. How to proceed with the next column when meeting the condition in 11.

13. How to place the answer in the sum when a column has been added and the total sum of these figures is 10 or more than 10.

14. How to proceed with the next column when meeting the condition in 13; i.e., carrying.

15. Ability to remember the number carried.

16. How to proceed when the need for carrying and no carrying is met alternately in an example.

17. How to place all the numbers in the sum.

18. How to check for correct answers.

2. Detailed Analysis of the Basic Number Facts Used.

There are one hundred basic number facts in addition which the child must be taught. Weakness in knowledge of these number facts results in inaccurate work in the application of the addition process in the solution of examples. In making a diagnosis in the primary grades, one of the first steps should be to determine the pupil's knowledge of the hundred basic combinations. This may be determined by finding how rapidly and how accurately he can give the answers to sets of examples such as the following, in which the number facts are grouped according to their approximate degree of difficulty.

The answers should be given orally for sets 1 and 2 in one minute, and for sets 3 and 4 in one and one-half minutes. The examiner should note the combinations whose sums are given incorrectly and assign these facts for further study.

Practice in Addition: Set I

5	1	3	0	3	1	1
1	0	1	5	0	5	3

0	4	6	2	4	2	0
1	2	0	4	1	2	6

4	3	0	1	0	1	2
0	2	4	2	3	4	1

0	5	3	1	2	2	0
0	0	3	1	0	3	2

Practice in Addition: Set II

8	1	2	1	6	0	3	4	7
0	9	6	8	1	7	5	6	2

5	9	6	4	4	4	3	0	5
2	1	2	5	3	4	7	9	5
7	6	1	8	2	0	1	2	9
1	3	6	2	7	8	7	5	0
3	8	6	5	3	7	7	2	5
4	1	4	3	6	3	0	8	4

Practice in Addition: Set III

4	6	3	9	2	5	3	4	2	7
7	6	5	3	8	7	8	6	9	3
3	7	2	6	8	3	6	3	4	3
9	4	7	4	3	6	5	7	8	4
9	1	7	2	5	6	1	4	8	5
2	9	5	6	6	3	8	5	4	3

Practice in Addition: Set IV

5	8	6	9	7	4	9
9	8	7	9	8	9	8
8	9	6	7	8	5	7
7	4	8	6	9	8	9
8	9	8	9	7	9	6
6	7	5	6	7	5	9

3. Skills in Column Addition.

(a) Addition by endings, including bridging the tens. An analysis of the work done by a pupil in solving such an addition example as that at the right reveals at once the skills the pupil must acquire in order to be able to work

95
56
68
89

the example correctly. For example, if addition is begun from the top of the column the pupil first says 11; then he adds 8 to 11, saying 19; then he adds 19 and 9, saying 28. He writes 8, and carries the 2. In like manner he continues in the second column until the example is completed.

$$\begin{array}{r} 11 \\ +8 \\ \hline \end{array}$$

In giving the sum of 11 and 8, addition by endings is introduced as a new difficulty. The difficulty is complicated by the fact that the number, 11, is not seen but is "thought of" mentally. The same is true for the addition of 19 and 9, since 19 is not seen, but is "thought of" mentally.

$$\begin{array}{r} 19 \\ 9 \\ \hline \end{array}$$

To find the sum of 19 and 9, the new difficulty of bridging the tens is introduced, a step which causes much difficulty for pupils.

$$\begin{array}{r} 19 \\ 9 \\ \hline \end{array}$$

If it is assumed that the pupils should be given practice on all the number combinations involving addition by endings, a very large number of practice exercises would need to be prepared. For example, to any number, such as 47, any number from 0 to 9, ten numbers in all, may be added. In order to provide for practice in adding by endings of all unit digits, 0 to 9, to all numbers less than 100, not including the 100 basic facts, 10×90 or 900 combinations would need to be practiced.

There is no experimental evidence available which would justify the conclusion that all of these 900 combinations must be practiced. Curriculum investigations have shown that long addition examples involving sums as large as 60 or 70 in the addition of columns of figures occur very rarely in life. It has therefore appeared

adequate to limit the work in addition by endings largely to sums of less than 40 or 50.

In order to discover possible weakness in addition by endings the following sets of examples may be used. It should be pointed out that the fact that a pupil can give the sums correctly when the numbers are presented to him in this form is no guarantee that he can transfer the skills involved directly to the addition of columns of numbers, since column addition is a much more complicated process.

ADDITION BY ENDINGS—NO BRIDGING. ($1\frac{1}{2}$ minutes)

$23 + 4 =$	$17 + 2 =$	$26 + 3 =$	$14 + 4 =$	$24 + 5 =$
$36 + 2 =$	$25 + 2 =$	$12 + 5 =$	$27 + 2 =$	$33 + 5 =$

42	18	27	3	4	8	26	35
6	1	2	35	22	31	3	4

Less than 16 correct in the time allowed indicates possible difficulty.

ADDITION BY ENDINGS—BRIDGING. ($1\frac{1}{2}$ minutes)

$26 + 7 =$	$18 + 6 =$	$46 + 4 =$	$23 + 8 =$	$42 + 8 =$
$17 + 9 =$	$35 + 8 =$	$33 + 9 =$	$29 + 7 =$	$44 + 9 =$

38	47	33	45	6	8	5	9
9	4	8	8	47	27	17	12

Less than 16 correct in the time allowed indicates possible difficulty. The pupil should then be given additional practice on similar examples.

(b) **Carrying.** As has been indicated, carrying in column addition presents many difficulties to pupils. The following examples illustrate some of the skills that are involved:

(*a*)	486	251	
	504	226	Carrying in unit's place.
		218	
(*b*)	253	154	
	492	283	Carrying in ten's place.
		521	
(*c*)	257	598	
	495	176	Carrying in unit's and ten's place.
		169	
(*d*)	5426	5158	
	2637	2918	Carrying in alternate places.
		1918	
(*e*)	506	306	
	208	208	Carrying into zero.
		209	
(*f*)	584		
	26		Carrying to a vacant place.
	2		
(*g*)	598		
	2		Carrying in adding by endings.

The essential skills involved in carrying are: 1. Knowing what number to write in the sum; 2. Knowing what number to carry; 3. Knowing the process to be used in the completion of varied types of examples involving carrying. To the adult these skills seem so simple and

obvious that the difficulties which pupils encounter in solving the examples involving carrying are not recognized. In making a diagnosis of pupil difficulties in addition, the mental processes of the child in carrying must be carefully studied to determine his habits of work while solving the example. Correct habits of carrying must be established from the beginning to insure efficiency in addition.

Carefully prepared practice exercises containing the various types of skills involved in carrying should be prepared. One new skill should be introduced at a time. Other skills should be added as rapidly as the pupils become efficient in their work.

A diagnostic test in addition of integers is given on page 118. This test contains the important types of examples in addition, each involving a different combination of skills or a special difficulty.

In giving this test to pupils in grades 3 and 4, the teacher should study carefully the work of each pupil to determine, if possible, the types of examples with which he has difficulty. If the cause of the difficulty is not obvious from the survey of the work on the paper, the pupil should be required to repeat orally the steps in solving the example, so that faulty procedures may be discovered.

4. Addition Involved in Multiplication.

In any multiplication example with carrying, addition combinations involving adding by endings are used, hence the pupil should be given sufficient drill on the different combinations that he becomes proficient in all the skills in addition. These will be discussed in connection with multiplication.

Finding Difficulties in Addition[1]

This exercise contains many different kinds of addition examples. Practice until you can work all of them correctly.

Set I

	a	b	c	d	e
1.	18 7	4 6 18	5 1 5	9 6 2 1	18 51
2.	753 236	5,241 4,537	13 21 42	342 104 452	30 20 20
3.	204 301 203	81 26 32	7,104 2,060	46 3 30	304 72 3
4.	58 24	627 258	3,965 3,714	5,248 2,396	4,953 7,638

Set II

	a	b	c	d	e
5.	8,564 1,987	68 47 53	337 406 238	162 360 407	957 777 995
6.	3,926 2,518 6,029	807 508 808	926 565 409	298 46 785	399 7 607
7	$.01 .04 .08	$.60 .40 .20	$.25 .54 .72	$1.65 2.64 7.28 6.53	$4.89 .26 .05 8.08

5. Difficulties in Addition of Integers.

After it has been determined by a test that a pupil is deficient in addition as measured by either his speed or

[1] Triangle Arithmetics, Book II, Part I, page 17.

accuracy of work, the next step is to determine the causes of the deficiency and the desirable remedial exercises. Many pupils make satisfactory progress by the use of well constructed, standardized practice exercises; others make relatively little, if any progress, and present serious problems for the teacher.

A careful study of the mental processes of pupils who are deficient in arithmetic reveals many faulty habits, which, unless corrected, result in inaccurate, inefficient work in addition. Table 2 shows the relative frequency of the most common faults in addition. It is based on a rearrangement of data from the investigation by Buswell and John.[1]

The data in Table 2 shows that in grades 3 to 6 the most common source of difficulty in addition for the 414 pupils whose work was studied was errors in addition combinations. The next most common faulty habit was counting. Pupils count with their fingers, tongue, toes, by nodding their heads, by tapping with their pencils, and in many other ways. Undoubtedly the weaknesses and faults due to errors in combinations and counting are the result of ineffective work in the lower grades. The remedy for these difficulties is more work on the basic number facts themselves rather than on difficult examples in column addition. The diagnostic methods described on pages 62 to 72 show how it is possible to determine the specific deficiencies and to do the necessary remedial work.

The largest group of faulty habits in addition involves carrying. Many of the pupils added the carried number

[1] G. T. Buswell and L. John, "Diagnostic Studies in Arithmetic," *Supplementary Educational Monograph*, No. 27 (Chicago: University of Chicago, 1926), p. 136.

TABLE 2

Frequency of Faulty Habits in Addition
(Adapted from Buswell and John)

	Grades				Total
	III	IV	V	VI	
1. Errors in Combinations	81	103	78	58	320
2. Counting	61	83	54	17	215
3. Carrying:					
(a) Added carried number last	39	45	45	26	155
(b) Forgot to add carried number	37	38	34	17	126
(c) Added carried number irregularly	26	30	28	18	102
(d) Wrote number to be carried	34	25	18	12	89
(e) Carried wrong number	28	19	26	14	87
(f) Carried when nothing to carry	6	9	9	5	29
(g) Wrote carried number in answer	10	2	2	1	15
(h) Added carried number twice	0	1	0	0	1
(i) Subtracted carried number	0	0	0	1	1
4. Faulty procedure:					
(a) Retraced work partly done	26	34	39	22	121
(b) Irregular procedure	16	29	23	18	86
(c) Grouped numbers	25	22	21	16	84
(d) Split numbers	12	29	25	14	80
(e) Lost place in column	17	17	17	14	65
(f) Disregards column	34	11	9	1	55
(g) Omits digits	13	21	13	5	52
(h) Disregarded one column	15	11	8	2	36
(i) Error in writing answer	12	3	14	5	34
(j) Added in pairs	6	6	6	2	20
(k) Added same digit in two columns	10	6	1	1	18
(l) Began with left column	1	1	1	0	3
5. Lapses, and other miscellaneous faults:					
(a) Used wrong operation	23	25	20	11	79
(b) Depended upon visualization	24	8	27	2	61
(c) Errors in reading numbers	14	10	21	7	52
(d) Dropped back one or more tens	13	12	17	5	47
(e) Derived unknown from known	13	7	11	11	42
(f) Skipped decades	11	7	9	5	32
(g) Confused columns	1	0	0	0	1
(h) Added imaginary numbers	0	0	1	0	1
6. Used scratch paper	7	5	9	0	21
Total cases	96	124	116	78	414

last, instead of at the beginning of a new column, resulting in many errors due to forgetting what number was to be carried. This fault can be corrected by showing the pupil the desirability of adding the carried number first, and then giving practice until the habit is established. Other pupils forgot to add the carried number at times; some carried the number irregularly, sometimes at the beginning of the column, sometimes in the middle, and sometimes at the end; some pupils carried the wrong number; some carried when there was no number to carry; some wrote the carried number as a "crutch" to aid them to remember the number. Other peculiar methods of carrying are given in the table. The data make obvious the necessity of giving special attention to the habits that are formed when carrying is being taught. The data also show the types of faults in carrying that the teacher would expect to discover in the work of pupils who are deficient in addition.

Faulty procedures are often discovered in the work of deficient pupils. Some pupils have no regular procedure in working addition examples. They begin at the top or the foot of the column; they add the large numbers first and then the small ones, or vice versa; they attempt to group numbers by tens, and in other ways; they split numbers to get easier combinations; they lose their place in a column, possibly due to lapses in attention or because they are easily distracted by environmental influences such as noises, whispering, and the like. Numerous other types of faulty procedures are given in the table. The following description of the methods of work used by pupils deficient in addition is contained in a report by Uhl:[1]

[1] W. L. Uhl, *loc. cit.*

Pupils' methods of work.—The findings as to methods employed by pupils in "difficult" combination are both interesting and significant. The following methods were found in the work of pupils who were tried out in the manner just described. A fourth-grade boy showed by slow work that the combination $9 + 7 + 5$ was difficult for him. When questioned, he showed that he used a common form of "breaking up" the larger digits. In working the problem, he said to himself: "$9 + 2 + 2 + 2 + 1 = 16$ and 21." This shows that the $9 + 7$ combination was not known but that the $16 + 5$ combination was known inasmuch as he arrived at "21" directly after having combined the other two numbers. Another boy of the same grade showed the same type of difficulty in a more pronounced form. He added 8, 6, and 0 as follows: "First take 4, then take 2, then add 8 and 4—makes 12, and 2 makes 14." In adding 9, 7, and 5, he said: "9 and 3 is 12 and 4 is 16 and $2 = 18$; and $2 = 20$; and $1 = 21$." He broke into parts even so easy a problem as $3 + 4 + 9$, adding $9 + 3 + 2 + 2 = 16$.

It is not known to what extent pupils who are efficient in arithmetic have irregular methods of work which may reduce their efficiency below possibly higher levels. The fact that the faulty habits listed in the table were found in the work of deficient pupils suggests the importance of teaching the pupil from the beginning, effective, economical methods of work. In the past, the establishment of methods of attack has too often been left to chance, and the faults described above have resulted.

Various psychological factors such as lapses in attention, short attention span, confusion of processes, and the like are a prolific cause of difficulty in addition. Much of the difficulty in adding long columns of figures in grades 3, 4, and 5 is due to the breaks in attention as the pupil proceeds up or down a column of figures. For example, in adding the column of numbers on the next page the pupil

98
47
56
58
84
72
65

might have added 8, 7, 6, and 8, getting 29 as the sum, and then become confused in adding this number and 4, due to a break in attention. The writer has found many pupils who knew the combinations, knew adding by endings, bridging of the tens, and the carrying procedure, but who failed completely in adding a column of figures such as the above because they did not know how to proceed when the break in attention came in adding the numbers. The lower the ability of the pupil and the greater his lack of control over the basic facts, the more frequent will be his lapses in attention. The average attention span of pupils in grades 3 and 4 is much less than that of pupils in the upper grades. It is therefore probably best not to give pupils in grades 3 and 4 addition examples containing more than five numbers in a column. The length of the column can be increased in the upper grades to seven or eight numbers in a column.

Courtis[1] describes difficulty due to faulty attention as follows:

"Irregular speed in adding a column may be due to either of two factors: lack of control of attention, or lack of knowledge of the combinations. Attention will be considered here.

"There is a limit to the length of time that a person can carry on any mental activity continuously. As time goes on, the mind tends to respond more and more readily to any new mental stimulus than it does to the old. The mind wanders, as it is said. The attention span for many children is six additions; for some, only three or four; for others, eight, or ten, and so on. That is, a child whose attention span is limited to six figures may add rapidly, smoothly, and accu-

[1] S. A. Courtis, *Standard Practice Tests in Arithmetic*, Manual (Yonkers-on-Hudson, New York: World Book Company, 1916).

rately, for the first five figures in the column, giving its attention wholly to the work. As the limit of its attention span is reached, however, it becomes increasingly difficult for it to concentrate its attention. The child suddenly becomes conscious of its own physical fatigue, and of the sights and sounds around it. The mind balks at the next addition; it may be a simple combination, as adding 2 to the partial sum, 27, held in mind. It finally becomes imperative that the child momentarily interrupts its adding activity and attends to something else. If this is done for a small fraction of a second, the mind clears and the adding activity will go on smoothly for a second group of six figures. The inattention may be repeated.

"It should be evident that these periods of inattention are critical periods. If the sum to be held in mind is 27, there is great danger that it will be remembered as 17, 37, 26, or some other number, as the attention returns to the work of adding. The child must, therefore, learn to 'bridge' its attention spans successfully. It must learn to recognize the critical period when it occurs in order to divert its attention consciously while giving its mind to remembering accurately the sum of the figures already added. This is probably best done by mechanically repeating to one's self mentally, twenty-seven, twenty-seven, twenty-seven, or whatever the sum may be, during the whole interval of inattention. Little is known about the different methods of bridging the attention spans and it may well be that other methods would prove more effective. The use of the device suggested above, however, is common.

"Giving up in the middle of a column and commencing again at the beginning is almost a certain symptom of lack of control of the attention. On the other hand, mere inaccuracy of addition (as 27 plus 2 equals 28) may be due to lack of control over the combinations. If the errors occur at more or less regular points in a column, and if, further, the combinations missed vary slightly when the column is re-added, the difficulty is pretty sure to be one of attention and not one of knowledge."

B. Subtraction of Whole Numbers

1. Analysis of the Subtraction Process.

One of the most difficult processes in arithmetic is subtraction. An analysis[1] of the skills and knowledges needed in subtraction is as follows:

1. The 100 subtraction combinations.
2. Three ideas in one's subtraction concept:
   Taking away idea: 15 − 7; 7 from 15.
   Adding idea: What number added to 7 equals 15?
   Difference idea: 15 is how many more than 7?
3. The meaning of the following terms: Minus, less, subtrahend, minuend, borrowing, difference, remainder.
4. The meaning of the subtraction sign.
5. That the complete minuend must always be larger than the complete subtrahend.
6. That in writing the example, units must be placed under units, tens under tens, etc.
7. That one must begin at the right and work to the left.
8. That the order of units in the subtrahend must be subtracted from the same order in the minuend.
9. How to proceed when the first number to be subtracted in the minuend is larger than the corresponding number in the subtrahend.
10. That one must not borrow unless the number in the subtrahend is larger than the corresponding number in the minuend.
11. How to proceed when a number of the subtrahend is larger than the corresponding number of minuend; i.e., borrowing.
12. What it means to place a 1 in front of a number when borrowing ten.

$$\begin{array}{r} 423 \\ -219 \\ \hline \end{array}$$

[1] E. Merton, *loc. cit.*

13. What it does to the next number in the minuend when a 1 has been placed before the following number.

14. Must be able to remember the new number made through borrowing.

$$\begin{array}{r} 628 \\ -\ 239 \\ \hline \end{array}$$

After subtracting 9 from 18, the child is dealing with 11, not 12.

15. How to proceed when the need for borrowing and no borrowing is met alternately in an example.

16. How to borrow when two or more successive digits in the subtrahend are larger than the corresponding digits in the minuend.

17. How to proceed when there are fewer figures in the subtrahend than in the minuend.

18. How to proceed when the last subtraction takes place with the subtrahend and minuend the same:

$$\begin{array}{r} 649 \\ -\ 623 \\ \hline \end{array}$$

(The zero must not be placed in the remainder.)

19. Ability to handle a zero or a succession of zeros in the subtrahend.

20. Ability to handle a zero or a succession of zeros in the minuend.

21. How to check for correct answers.

A well organized teaching procedure must recognize the necessity of providing for the development of each of the elements included in the analysis. Deficiency in any element may be the cause of weakness in the process as a whole.

2. THE BASIC SUBTRACTION FACTS.

A knowledge of the basic facts in subtraction is absolutely essential for satisfactory work in that process. Since this is true the first step in a diagnosis should be to deter-

nine the control the pupil has over these facts. This should be done by measuring the speed and accuracy with which he can give orally or write the remainders for combinations and by discovering his deficiencies.

The following sets of combinations may be used to determine the pupil's knowledge of the basic facts. A pupil in grade 3 or 4 should be able to give the answers for each set in the time indicated. If he cannot do so, additional practice is needed to develop speed. Special work should be done on combinations not known.

PRACTICE IN SUBTRACTION: SET I (Two minutes)

3	6	2	3	4	6	5
3	0	1	1	3	4	0
1	4	4	3	6	0	5
0	2	1	2	6	0	4
4	5	2	6	4	6	1
0	3	2	3	4	5	1
2	5	6	5	3	6	5
0	2	2	1	0	1	5

PRACTICE IN SUBTRACTION: SET II (Two minutes)

8	9	8	7	7	10	9	8	8
1	4	8	2	4	1	9	4	2
7	10	9	9	9	7	10	10	7
3	3	1	8	0	6	6	5	0
8	8	8	9	10	9	10	7	9
5	7	3	3	4	6	7	1	7
7	10	7	10	9	10	9	8	8
7	8	5	2	2	9	5	0	6

PRACTICE IN SUBTRACTION: SET III (Two minutes)

11	7	10	11	9	7	12	8
4	3	6	9	3	4	5	5
12	10	11	10	12	10	9	12
3	3	8	4	6	7	2	8
9	11	9	12	10	11	10	12
4	3	7	9	8	7	9	4
11	8	11	9	11	12	8	10
6	3	2	6	5	7	6	2

PRACTICE IN SUBTRACTION: SET IV (Two minutes)

15	14	16	13	18	13	14
6	8	7	5	9	6	9
17	16	13	14	15	13	16
8	9	4	6	8	7	8
15	13	14	17	13	14	15
7	9	7	9	8	5	9

3. HIGHER DECADE SUBTRACTION.

Whenever a pupil works a short division example of the type, $4\overline{)1956}$, higher decade subtraction is used. For example, after dividing 19 by 4, the subtraction example, 19 – 16, must be worked mentally; the 3 must be carried to the 5, forming 35. Here the example 35 – 32 must be worked mentally after division by 4. Similar types of subtraction examples must be worked in every short division example involving carrying. Specific practice on such subtraction examples improves the work in short division. The detailed discussion of this point will be taken up in the unit on division of whole numbers.

4. Examples not Involving Borrowing or Carrying.

When compared with addition, subtraction is a much less complicated process as far as the basic number facts to be learned are concerned. Only two numbers are involved, namely, the minuend and the subtrahend, whereas in addition, long columns of as many as nine, or ten, or more numbers, are sometimes found.

The chief difficulties in examples not involving carrying or borrowing are found in examples which contain zero difficulties, as follows:

$\begin{array}{r} 84 \\ -24 \\ \hline \end{array}$		A number less the number equals zero.
$\begin{array}{r} 65 \\ -20 \\ \hline \end{array}$		A number less zero equals the number.
$\begin{array}{r} 47 \\ -45 \\ \hline \end{array}$	$\begin{array}{r} 598 \\ -592 \\ \hline \end{array}$	Zero is not written when occurring in this type.
$\begin{array}{r} 85 \\ -4 \\ \hline \end{array}$	$\begin{array}{r} 487 \\ -6 \\ \hline \end{array}$	How to subtract in examples involving a larger number of digits in the minuend than in the subtrahend.

Examples of the type below merely afford practice on the combinations, although care must be taken to establish the habit of beginning work at the right.

$$\begin{array}{r} 976 \\ -432 \\ \hline \end{array} \qquad \begin{array}{r} 800 \\ -400 \\ \hline \end{array} \qquad \begin{array}{r} 906 \\ -804 \\ \hline \end{array} \qquad \begin{array}{r} 6892 \\ -4571 \\ \hline \end{array}$$

5. Examples Involving Borrowing or Carrying.

After the pupil has learned the basic combinations, and has learned to work examples such as those given under

(4), he must learn the process of working examples in which the number in the minuend is less than the number in the subtrahend. There are many skills that must be taught, the chief ones being illustrated by the following examples:

$\begin{array}{r}158\\ -\ 75\\ \hline\end{array}$	Where the combination $\begin{array}{r}15\\ -\ 7\\ \hline\end{array}$ introduces the process.
$\begin{array}{r}95\\ -\ 28\\ \hline\end{array}$	Borrowing or carrying in unit's place.
$\begin{array}{r}60\\ -\ 47\\ \hline\end{array}$	Zero difficulty.
$\begin{array}{r}73\\ -\ 64\\ \hline\end{array}$	Zero difficulty in $\begin{array}{r}6\\ -\ 6\\ \hline\end{array}$
$\begin{array}{r}63\\ -\ 9\\ \hline\end{array}$	Subtraction by endings; blank place.
$\begin{array}{r}962\\ -\ 138\\ \hline\end{array}$	Three numbers; borrowing or carrying in unit's place.
$\begin{array}{r}708\\ -\ 495\\ \hline\end{array}$	Borrowing or carrying ten's place.
$\begin{array}{r}845\\ -\ 269\\ \hline\end{array}$	Borrowing or carrying in unit's and ten's place.
$\begin{array}{r}435\\ -\ 86\\ \hline\end{array}$	Two-place borrowing or carrying; blank place.
$\begin{array}{r}600\\ -\ 497\\ \hline\end{array}$	Double zeros in minuend.
$\begin{array}{r}207\\ -\ 29\\ \hline\end{array}$	Zero in ten's place; borrowing or carrying.

$$\begin{array}{r} 800 \\ -\ 208 \\ \hline \end{array}$$ Double zeros in minuend; borrowing; zero in subtrahend.

These examples may be followed by four-place or five-place numbers, involving other combinations of difficulties. The diagnostic test on this page contains many different types of subtraction examples, each type representing a new combination of skills. Such a test makes it possible for the teacher to locate the specific types of examples which cause the pupil difficulty and to prescribe the necessary remedial exercises.

Finding Difficulties in Subtraction[1]

This exercise contains many different kinds of subtraction examples. Practice until you can work all of them correctly.

Set I

	a	*b*	*c*	*d*	*e*
1.	9 4	15 7	87 3	46 21	897 356
2.	9 9	25 5	258 101	90 30	805 202
3.	77 72	686 683	869 22	774 414	793 593

Set II

	a	*b*	*c*	*d*	*e*
4.	73 8	494 49	868 539	729 456	921 656
5.	6,172 4,753	523 195	180 2	909 727	802 618

[1] Triangle Arithmetics, Book II, Part I, p. 24.

6.	4,001 2,883	9,200 3,373	574 108	4,981 3,008	8,043 7,606

Set III

7.	9,080 6,759	8,010 3,584	11,900 9,901	7,462 4,399	3,376 2,968
8.	15,886 5,899	16,575 6,778	2,333 1,677	12,344 4,446	6,661 5,892
9.	$.08 .03	$1.20 .80	$4.00 1.25	$6.54 5.79	$8.91 4.06

6. The Most Common Faulty Habits in Subtraction. An analysis of the faulty habits in subtraction revealed by a study of the mental processes of pupils deficient in that process is given in Table 3.

TABLE 3

Frequency of Occurrence of the Most Common Faults in Subtraction of Whole Numbers

(Adapted from Buswell and John, page 137)

	Grades				Total
	III	IV	V	VI	
1. Errors in Combinations	62	75	69	40	246
2. Borrowing:					
(a) Did not allow for having borrowed	19	50	57	36	162
(b) Errors due to zero in minuend	25	39	26	15	105
(c) Subtracted minuend from subtrahend	47	33	12	4	96
(d) Failed to borrow; gave zero as answer	21	20	14	4	59
(e) Deducted in minuend when no borrowing was necessary	2	8	10	5	25
(f) Deducted (2) from minuend after borrowing	1	5	8	6	20
(g) Increased minuend digit after borrowing	2	2	6	2	12
(h) Deducted all borrowed numbers from left hand digit	1	0	1	0	2

TABLE 3 (*Continued*)

	Grades III	IV	V	VI	Total
3. Counting	43	44	39	10	136
4. Faulty Procedures:					
(*a*) Said example backward	21	38	29	12	100
(*b*) Added instead of subtracted	18	9	19	1	47
(*c*) Used same digit in two columns	18	15	3	4	40
(*d*) Omitted a column	9	13	8	5	35
(*e*) Split numbers	7	5	10	2	24
(*f*) Ignored a digit	12	6	2	3	23
(*g*) Used minuend or subtrahend as remainder	10	6	2	0	18
(*h*) Began at left column	2	0	1	0	3
5. Lapses, etc.:					
(*a*) Derived unknown from known	12	9	13	3	37
(*b*) Error in reading	14	5	13	10	42
(*c*) Error due to numbers in minuend and subtrahend being the same	1	5	10	3	19
(*d*) Reversed digits in remainder	4	7	2	4	17
(*e*) Confused process with division or multiplication	5	6	3	2	16
(*f*) Skipped one or more decades	3	4	7	0	14
(*g*) Based subtraction on multiplication combination	1	2	3	0	6
(*h*) Error in writing answer	2	1	0	1	4
Total cases	84	109	109	70	372

Errors in the basic subtraction combinations and counting to get the answer are the most common faults in subtraction. Their prevalence suggests the need of additional work to develop speed and greater accuracy in the basic number facts. This practice should be adjusted to the needs of the individual by careful pre-tests and an analysis of the results.

The skills involved in borrowing or carrying present the largest group of faults in subtraction. The most common fault in this group was failure of the pupil to allow for having borrowed. Inability to solve examples

with zeros in the minuend caused many errors. This suggests the need of special stress on examples involving zeros in the minuend. Complete lack of comprehension of the process involved is evidenced by the individuals who subtracted the smaller number in the minuend from the larger number in the subtrahend, or who failed to borrow when necessary and gave zero for the answer. Most of these faults can be eliminated by a careful reteaching of the borrowing or carrying process by means of a well graded set of practice exercises each of which presents a new difficulty.

Various faulty procedures are found in the work of deficient pupils. Faulty statements, splitting up numbers, omitting numbers, and using the same number in two columns were those most frequently discovered. The following is an exact reproduction of the steps in solving a subtraction example orally, reported by Winch:[1]

(1) "9 from 6 you cannot; take 1 from the 7 next door leaves 6; 9 from 10 is 1 and 6 is 7."

(2) "2 from 6 leaves 4."

(3) "8 from 5 you cannot; take 1 next door, leaves 3; 8 from 10 leaves 2, and 5 are 7."

(4) "9 from 3 you cannot; go next door, take 1 leaves 1; 9 from 10 is 1, and 3 makes 4."

$$\begin{array}{r} 624{,}576 \\ 139{,}829 \\ \hline 484{,}747 \end{array}$$

(5) "3 from 1 you cannot; go next door, take 1 from the 6 leaves 5; 3 from 10 is 7 and 1 makes 8."

(6) "2 from 6 leaves 4."

Similar statements of the pupil's verbal processes are frequently found in educational literature. They illustrate the complexity of the mental processes employed by

[1] W. H. Winch, "Equal Addition Versus Decomposition in Teaching Subtraction," *Journal of Experimental Pedagogy* (Vol. 5, pp. 207–20, and 261–70).

many pupils in arriving at the solution of an example. Such roundabout methods are inefficient and cumbersome and result in inferior work. One fifth-grade pupil was given the example at the right to solve. After laboring for a time he finally wrote the correct answer, 47. After considerable questioning it was revealed that he had found the answer by counting from 84 back to 37 by ones and had managed, in some manner, to keep track of the count and to get the correct answer. He simply did not know what to do when the number in the minuend was less than the corresponding number in the subtrahend. It may be said that he readily learned the correct process and was greatly pleased to see how much more quickly and easily he could find the answer than by the method he had been using.

$$\begin{array}{r} 84 \\ -37 \\ \hline \end{array}$$

C. Multiplication of Whole Numbers

1. Basic Knowledge in Multiplication.

According to Merton[1], the basic knowledge needed in multiplication is as follows:

1. The multiplication tables through 9 × 9 including zeros.
2. How to add.
3. The meaning of the multiplication sign.
4. The meaning of the following terms: Multiplication, product, multiplicand, multiplier, carrying, and sum.
5. That in writing the example, units must be placed under units, tens under tens, etc.
6. That the multiplier is always a number of *times*.
7. That the number in the multiplicand is to be multiplied by the numbers in the multiplier.

[1] Merton, *loc cit.*

8. That one must begin at the right and work to the left.

9. How to place the product after the first multiplication when the product is less than 10.

10. Ability to proceed with the multiplication of the next digit when meeting the condition in 9.

11. How to place the answer in the product when the product is 10 or more than 10.

12. How to proceed with the multiplication of the next digit when meeting the condition in 11; i.e., *carrying*.

13. Ability to *multiply and add* quickly.

14. Ability to remember the number carried.

15. Ability to handle zero or a succession of zeros in the multiplicand.

16. How to proceed when the need for carrying and no carrying are met alternately in the example.

17. How to proceed after the multiplication by the units figure of the multiplier is completed when there is more than one figure in the multiplier.

(a) Which number to multiply by next.

(b) Where to place the first product.

18. Ability to handle a zero or a succession of zeros in the multiplier.

19. That these products must be added and how this is done. This involves any or all of the 18 points on addition.

20. How to check for correct answers.

2. The Basic Number Facts.

(a) **Zero combinations.** Unless they are carefully presented, combinations, one of whose factors is zero, have been found to be difficult for children. There are 19 such combinations. When being taught they should be practiced in connection with the other basic facts.

(b) **Combinations not involving zero.** Assuming that the pupil should know all combinations neither of whose factors exceeds 9, there are in all 81 facts to be learned, not including zeros. There are 100 facts in all.

The following sets of combinations provide a convenient means of locating deficiencies in multiplication facts. Pupils in grades 4 and 5 should be able to give all of the facts in each set in the time allowed. Special note should be made of the facts on which the pupil makes mistakes. Special work on them should be required.

SET I. 1 minute.

1	2	1	2	1	2	1	2
6	0	1	3	2	4	7	2
1	1	2	1	1	0	1	1
3	8	1	4	9	0	0	5
2	2	2	2	0	0	2	0
7	9	5	8	1	4	6	2

SET II. 1½ minutes.

3	4	3	4	3	3	4
1	1	5	6	0	3	3
4	3	4	3	4	4	3
7	4	0	2	5	8	7
3	4	3	0	3	4	4
8	2	9	3	6	4	9

SET III. 1½ minutes.

5	6	5	6	5	5	6
1	6	2	1	6	3	7

6	5	6	5	6	6	6
3	4	5	5	4	8	0

5	6	5	5	6	5	0
8	2	7	0	9	9	5

Set IV. $1\frac{1}{2}$ minutes.

8	7	8	8	7	7	7
2	1	3	6	5	2	9

7	0	7	8	8	7	0
8	7	3	9	4	7	6

7	8	7	8	7	8	8
4	1	0	7	6	8	5

Set V. $1\frac{1}{2}$ minutes.

9	9	9	8	9	9
1	7	2	0	3	0

9	9	0	9	9	9	0
8	4	9	6	9	5	8

3. Carrying in Multiplication.

$$\begin{array}{r} 43 \\ \times 4 \\ \hline \end{array}$$

Whenever a pupil works an example of such a type as the one at the left carrying is involved. The pupil must learn (1) to write the correct number in the product; (2) to carry in his mind the number to be carried; (3) to add the carried number to the next partial product. This process seems simple to adults but for the pupil it is a quite complicated process to learn, one which causes much difficulty.

One method of reducing the difficulty of this process is to give the pupil special practice on the possible combinations that may arise in the carrying step. The following types of exercises will aid the examiner to discover possible weaknesses in carrying in multiplication:

SET I

$27 + 4 =$	$18 + 3 =$	$49 + 5 =$	$12 + 5 =$	$36 + 4 =$
$45 + 8 =$	$40 + 6 =$	$35 + 4 =$	$16 + 3 =$	$81 + 8 =$
$18 + 5 =$	$15 + 4 =$	$72 + 7 =$	$32 + 6 =$	$24 + 7 =$

SET II

$4 \times 7 + 3 =$	$9 \times 6 + 7 =$	$3 \times 7 + 2 =$
$6 \times 6 + 3 =$	$9 \times 4 + 8 =$	$5 \times 4 + 3 =$
$7 \times 5 + 6 =$	$8 \times 6 + 4 =$	$2 \times 9 + 1 =$
$8 \times 5 + 4 =$	$3 \times 4 + 1 =$	$4 \times 5 + 2 =$
$6 \times 8 + 5 =$	$9 \times 7 + 4 =$	$7 \times 7 + 6 =$

Set (I) includes the sums of some of the possible products and a number that might be carried in an example in multiplication. Set (II) includes multiplication combinations plus numbers that might be carried, giving practice in multiplying and adding a given number to the product. It should be noted that in exercises like Set (II) the number carried must always be at least one less than the number by which one is multiplying. An example such as $4 \times 3 + 9$, never should be used, since 9 is never carried in a multiplication example when the multiplier is 4. The largest number that ever could be carried when the multiplier is 4 is the 3 in the product of $4 \times 9 = 36$.

The product numbers which occur in examples involving carrying in multiplication are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32, 35, 36, 40, 42, 45, 48, 49, 54, 56, 63, 64, 72, 81. There are in all 175 possible addition combinations involving carrying in multiplication.

The most common types of examples in multiplication involving carrying are as follows:

(*a*) $\begin{array}{r} 16 \\ \times 5 \\ \hline \end{array}$ Carrying in ten's place; two-place product.

(*b*) $\begin{array}{r} 65 \\ \times 6 \\ \hline \end{array}$ Carrying in ten's place; three-place product.

(*c*) $\begin{array}{r} 216 \\ \times 6 \\ \hline \end{array}$ Carrying in ten's place.

(*d*) $\begin{array}{r} 321 \\ \times 8 \\ \hline \end{array}$ Carrying in hundred's place.

(*e*) $\begin{array}{r} 4162 \\ \times 3 \\ \hline \end{array}$ Carrying in alternate places.

(*f*) $\begin{array}{r} 3617 \\ \times 5 \\ \hline \end{array}$

4. Zeros in Multiplication.

Zeros cause much difficulty in multiplication. The most common types of zero difficulties in simple multiplication are as follows:

(*a*) $\begin{array}{r} 40 \\ \times 2 \\ \hline \end{array}$ Zero at end.

(*b*) $\begin{array}{r} 203 \\ \times 3 \\ \hline \end{array}$ Zero in middle, no carrying.

(*c*) $\begin{array}{r} 200 \\ \times 4 \\ \hline \end{array}$ Double zero.

(*d*) $\begin{array}{r} 206 \\ \times 6 \\ \hline \end{array}$ Middle zero; carrying.

(*e*) $\begin{array}{r} 4007 \\ \times 8 \\ \hline \end{array}$ Double zero; carrying.

(*f*) $\begin{array}{r} 4080 \\ \times 9 \\ \hline \end{array}$ Alternate zero; carrying.

If pupils are given the above examples to work, their deficiencies in working with zeros quickly become apparent.

5. Long Multiplication, Including Zeros.

When the pupil has once learned all of the steps in simple multiplication, he must be taught the more difficult complex processes of multiplying by two or more numbers, including numbers ending in zeros, or including zeros. The new skills involved here are:

1. Ability to place partial products correctly.
2. Ability to allow for zeros in the multiplier.
3. Ability to add partial products.

These abilities are involved in many different types of examples in multiplication. The diagnostic test for grade 5 on page 142 contains a large variety of types of examples by means of which the teacher can locate the specific types which the pupil is not able to solve because of lack of knowledge of the procedure to follow.

In most cases an examination of the paper will show the exact nature of the difficulty. Where it is not possible to discover the cause of the difficulty in this way the pupil should be required to work the example aloud.

6. Faulty Habits in Multiplication.

The relative frequency of faulty habits of work of pupils in grades 3 to 6 deficient in multiplication is given in Table 4.

Finding Difficulties in Multiplication[1]

This exercise contains many different kinds of multiplication examples. Practice until you can work all of them correctly.

Set I

	a	b	c	d	e
1.	7 8	23 3	20 4	71 8	812 4
2.	340 2	601 5	400 3	2,010 6	504 7
3.	16 7	87 5	615 6	851 8	657 9

Set II

	a	b	c	d	e
4.	32 21	412 21	230 22	612 30	312 400
5.	412 103	3,004 22	52 93	24 85	369 78
6.	849 490	708 796	9,005 465	9,080 823	143 596

Set III

	a	b	c	d	e
7.	628 705	625 404	7,859 968	4,685 157	7,865 2,500
8.	\$.75 8	\$1.20 5	\$.09 7	\$2.25 79	\$16.00 64
9.	\$15.20 80	\$.60 90	\$.05 48	\$30.10 275	\$524 76

[1] Triangle Arithmetics, Book II, Part I, page 42.

TABLE 4

The Most Common Faults in Multiplication of Whole Numbers

(Adapted from Buswell and John, page 138)

	Grades				Total
	III	IV	V	VI	
1. Combinations:					
(a) Errors in combinations	36	59	60	41	196
(b) Errors in single zero combinations, 0 as multiplier	11	20	23	27	81
2. Carrying:					
(a) Error in adding carried number	6	40	58	45	149
(b) Carried wrong number	5	28	40	22	95
(c) Forgot to carry	10	30	27	22	89
(d) Counted to carry	4	20	28	9	61
(e) Wrote carried number	8	16	14	9	47
(f) Error in carrying into zero	1	6	7	1	15
(g) Multiplied carried number	2	1	0	1	4
(h) Added carried number twice	0	1	1	0	2
(i) Carried when nothing to carry	0	0	1	0	1
3. Counting:					
(a) Counting to get combinations	15	11	9	5	40
(b) Repeated table	3	11	11	6	31
(c) Multiplied by adding	6	11	8	4	29
(d) Wrote table	0	0	4	1	5
4. Faulty Procedures:					
(a) Wrote rows of zeros	2	33	40	34	109
(b) Used multiplicand as multiplier	18	33	23	15	89
(c) Errors due to zero as multiplier	5	26	30	17	78
(d) Omitted digit in multiplier	1	15	20	16	52
(e) Omitted digit in multiplicand	2	17	12	12	43
(f) Errors due to zero in multiplicand	4	14	15	9	42
(g) Error in position of partial products	0	15	15	9	39
(h) Confused products when multiplier has two or more digits	1	13	9	9	32
(i) Did not multiply a digit in multiplicand	5	9	7	7	28
(j) Omitted digit in product	0	5	7	5	17
(k) Forgot to add partial products	0	3	7	2	12
(l) Split multiplier	0	1	6	4	11
(m) Wrote wrong digit of product	0	3	4	2	9
(n) Used multiplier or multiplicand as product	1	1	1	1	4
(o) Used digit in product twice	0	1	2	0	3
(p) Began at the left side	1	0	0	0	1
(q) Multiplied partial products	0	1	0	0	1

TABLE 4 (*Continued*)

	Grades				Total
	III	IV	V	VI	
5. Lapses, etc.:					
(*a*) Used wrong process	18	22	16	10	66
(*b*) Derived unknown combination from known	3	11	6	6	26
(*c*) Errors in reading	6	5	11	3	25
(*d*) Errors in writing products	2	4	8	2	16
(*e*) Multiplied twice by same digit	1	1	3	2	7
(*f*) Reversed digits in product	1	1	2	2	6
6. Errors in addition	5	31	41	21	98
7. Illegible figures	0	3	5	7	15
Total cases	47	98	102	82	329

The most common fault is lack of knowledge of the basic combinations especially those in which 0 is the multiplier. This fault is also revealed by the number of pupils who count to get the products of combinations or repeat tables. Evidently these pupils would profit greatly from further work on these basic facts.

Another important source of error in multiplication is in faulty carrying. Many errors are made in adding the carried number, showing the need of additional practice on the step of adding products and the number to be carried. Many other errors in carrying are due to the fact that the wrong number is carried. Pupils often forget to carry. They count with their fingers and in other ways when carrying, thereby revealing weakness in addition.

$$\begin{array}{r} 408 \\ \underline{\times 6} \end{array}$$

Errors are often made when carrying into zero in such an example as the one at the right. It is evident that teachers must give special

attention to the carrying process. In making an analysis of the causes of difficulty in multiplication, the pupil's habits of work in carrying should be carefully checked.

Numerous faulty procedures are found in examples involving the work in multiplying by two or more numbers especially when the multiplier contains one or more zeros. Many of these errors are probably due to carelessness, such as the omission of a digit in the multiplier or in the multiplicand; incorrect placing of the partial products; failure to complete the example, or using a digit in the multiplier twice.

An analysis of the data concerning faulty procedures suggests:

(1) The necessity of giving special attention to examples involving zeros in either the multiplier or multiplicand.

(2) The necessity of insisting on neat work and the correct placing of the partial products.

(3) The desirability of requiring the pupil to recheck the entire work before being prepared to accept the answer as correct.

Errors in the addition of partial products are also frequently found and constitute one of the chief causes of weakness in multiplication.

D. Division of Whole Numbers

1. The Knowledge Needed in Short Division.

According to Merton[1] the basic knowledge and abilities necessary in short division are as follows:

1. The multiplication tables through 9×9 including zeros.

[1] Merton, *loc. cit.*

2. The 100 subtraction combinations.
3. The division tables through the 9's including zeros.
4. The meaning of the signs for division, $\overline{)\quad\quad}$, and $\div$.
5. The meaning and use of each of the following terms: Division, product, dividend, carrying, divisor, quotient, remainder.
6. How to proceed when the first number of the dividend is the same as, or greater than, the divisor.
7. Where to place the first number of the quotient when meeting the condition in 6.
8. Must know how to proceed when the first number of the dividend is smaller than the divisor, as: $4\overline{)104}$.
9. Where to place the first number in the quotient when meeting the condition in 8.
10. Each step:
    (a) Divide.
    (b) Place quotient figure.
    (c) Multiply.
    (d) Subtract.
    (e) Carry.
11. How to handle the remainder after subtracting:

$$2\overline{)9154}.$$

12. Ability to remember the number carried.
13. That no number equal to or larger than the divisor can be carried.
14. Ability to find the correct quotient figure with a minimum of trial; i.e., rapid recognition of the two factors, one being given.
15. How to continue dividing after some number in the quotient brings no remainder:

$$\begin{array}{r} 18 \\ 2\overline{)364} \end{array}$$

16. Ability to handle the zero in the quotient and make the proper operation in the dividend:

$$\begin{array}{r} 150 \\ 4\overline{)6012} \end{array}$$

17. Ability to handle the zero in the dividend when alone or used with a number carried.
18. How to handle a zero or a number of zeros at the end of the dividend.

19. How to place correctly all quotient figures.
20. How to handle the remainder at the end of a problem that does not "come out even."
21. How to check for correct answers.

2. The Additional Basic Knowledge Needed in Long Division.

In addition to the knowledge listed for short division, the following knowledge is basic in the long division process:

1. Ability to subtract under conditions found in the long division process.
2. Ability to estimate quotients of all types, chiefly in examples with two- and three-place divisors.
3. How to multiply and carry.
4. The steps in the process.
   (1) Divide.
   (2) Place quotient figure correctly.
   (3) Multiply divisor and correct quotient figure.
   (4) Place the resulting product correctly.
   (5) Compare with number from which it is being subtracted.
   (6) Subtract.
   (7) Compare remainder and divisor.
   (8) Bring down digit from dividend.
   (9) Continue these steps until the example is completed.

If the correct basic habits in short division relating to the manipulation of zero difficulties and remainders are well established, the chief problems involved in long division are teaching the pupil the form to be used in writing out the example and how to find the quotient figures. The form in which to write out the work is easily taught by requiring the pupil to write out the steps in

solving a typical example in short division which the pupil has learned to solve mentally.

3. The Division Combinations.

(a) **The basic combinations.** In division there are 90 basic combinations with divisors from 1 to 9 and in which there is no remainder. Since these are absolutely essential any deficiency in knowledge of basic combinations is certain to have serious consequences. The following sets of examples afford a convenient means of checking up on the pupil's knowledge of the basic combinations:

Set I. 1 minute.

1)6	2)4	2)10	1)9	1)3	1)5
2)12	1)7	1)1	2)14	2)8	2)18
2)2	1)2	2)6	1)8	2)16	1)4

Set II. $1\frac{1}{2}$ minutes.

3)3	4)8	3)6	4)20	4)12	3)9
4)24	3)12	4)4	3)24	4)28	4)16
3)21	4)32	3)27	4)36	3)15	3)18

Set III. $1\frac{1}{2}$ minutes.

5)10	6)24	5)5	6)36	6)6	5)15
6)18	5)25	6)12	6)48	5)20	5)45
5)30	6)30	5)35	6)54	5)40	6)42

SET IV. $1\frac{1}{2}$ minutes.

$7\overline{)28}$	$8\overline{)24}$	$7\overline{)14}$	$8\overline{)40}$	$7\overline{)35}$	$8\overline{)48}$
$8\overline{)32}$	$7\overline{)7}$	$8\overline{)8}$	$7\overline{)56}$	$8\overline{)56}$	$7\overline{)49}$
$7\overline{)21}$	$8\overline{)16}$	$7\overline{)42}$	$8\overline{)64}$	$7\overline{)63}$	$8\overline{)72}$

SET V. 1 minute.

$9\overline{)45}$	$9\overline{)72}$	$9\overline{)9}$	$9\overline{)54}$	$9\overline{)27}$
$9\overline{)63}$	$9\overline{)18}$	$9\overline{)36}$	$9\overline{)81}$	

(b) **Combinations with remainders.** In dividing 13 by 2, the pupil must first find the quotient figure 6, then subtract 12 from 13 to get the remainder 1. It is known that pupils have much difficulty with this step since higher decade subtraction is involved and the pupil has usually had relatively little, if any, practice in this type of subtraction. Therefore, in addition to the 90 basic division facts, the pupil must have definite practice on 360 other division combinations in which there are remainders. This is a total of 450 division facts. These facts may be found as follows:

	TOTAL POSSIBLE
One into any number from 0 to 9	10
Two into any number from 0 to 19	20
Three into any number from 0 to 29	30
Four into any number from 0 to 39	40
Five into any number from 0 to 49	50
Six into any number from 0 to 59	60
Seven into any number from 0 to 69	70
Eight into any number from 0 to 79	80
Nine into any number from 0 to 89	90
Total	450

The following sets of exercises should be used to discover possible weakness in division examples involving remainders. The sets contain a fair sampling of combinations ranging from easy to difficult.

Set I. Subtraction Involved in Division.

$1\frac{1}{2}$ minutes.

52 − 49 =	17 − 9 =	19 − 18 =	44 − 36 =
67 − 63 =	15 − 12 =	31 − 24 =	48 − 42 =
52 − 48 =	27 − 21 =	29 − 25 =	52 − 45 =
19 − 15 =	20 − 14 =	35 − 27 =	39 − 36 =
15 − 12 =	14 − 10 =	34 − 28 =	17 − 12 =
24 − 20 =	26 − 18 =	39 − 32 =	23 − 16 =
32 − 28 =	27 − 21 =	41 − 35 =	61 − 54 =

Set II. Finding Remainders and Products.

2 minutes.

9 × 5 and ? = 48; 52	8 × 5 and ? = 47; 42
9 × 7 and ? = 67; 70	8 × 6 and ? = 51; 54
9 × 8 and ? = 77; 79	7 × 7 and ? = 52; 55
4 × 6 and ? = 27; 25	6 × 9 and ? = 57; 59
5 × 7 and ? = 38; 39	3 × 8 and ? = 26; 25

Set III. Easy Divisions and Remainders.

2 minutes.

(Subtraction in Same Decade)

2)15	3)17	4)14	3)23	2)7	4)19
4)35	5)47	4)29	5)4	4)38	5)23
4)8	5)24	5)39	2)17	3)19	5)46
2)19	3)26	4)21	5)18	4)37	4)27

SET IV. DIVISION AND REMAINDERS. $2\frac{1}{2}$ minutes.
(Higher Decade Subtraction)

$4\overline{)31}$	$6\overline{)21}$	$7\overline{)33}$	$4\overline{)11}$	$3\overline{)20}$	$6\overline{)41}$
$7\overline{)62}$	$8\overline{)53}$	$6\overline{)23}$	$8\overline{)62}$	$7\overline{)61}$	$9\overline{)44}$
$7\overline{)54}$	$9\overline{)60}$	$8\overline{)21}$	$6\overline{)53}$	$7\overline{)20}$	$9\overline{)80}$
$6\overline{)53}$	$8\overline{)31}$	$7\overline{)33}$	$9\overline{)62}$	$9\overline{)35}$	$7\overline{)41}$

4. TYPES OF SHORT DIVISION EXAMPLES.

The chief difficulties in short division examples are concerned with remainders in the body of the example or at the end, and with the use of zeros in the quotients. The following set of examples contains the important types of examples in easy short division. This set may be used to determine deficiencies in particular types of processes.

(*a*) $2\overline{)64}$ Even division, no remainders.

(*b*) $2\overline{)482}$ Even division; 3 place quotient.

(*c*) $2\overline{)608}$ Even division; zero in quotient in middle.

$2\overline{)420}$ Even division; zero in quotient at end.

$4\overline{)400}$ Even division; zero in quotient; double zero.

(*d*) $3\overline{)216}$ Initial trial dividend—two figures; no remainders.

(*e*) $2\overline{)816}$ Zero difficulty in quotient; carrying.

(*f*) $4\overline{)485}$ Remainders at end of example; with and without zero difficulty.

$3\overline{)905}$

$4\overline{)246}$

$5\overline{)254}$

(g) $2\overline{)819}$ Zero, carrying, and remainder.

(h) $3\overline{)282}$ Carrying; no remainder.

$4\overline{)96}$

(i) $4\overline{)347}$ Carrying and remainder.

If the pupil has difficulty with any of the types listed above, the reason for the difficulty should be discovered. The following are known to be the chief causes of difficulty in short division:

1. Errors in division combinations.
2. Errors in carrying.
3. Zero difficulties.
4. Manipulation of remainders.

If the pupils have special difficulty with any particular type of example they should be given practice on others of the same type. These can easily be prepared by the teacher.

5. Naming Quotients in Long Division.

One of the chief problems in long division is to teach the pupil effective techniques for finding the quotient figures. The difficulty of finding a quotient figure may vary from the easy type $20\overline{)60}$, to the very difficult trial division type, $16\overline{)142}$. The following sets of examples afford a convenient basis for determining the level at which the pupil has difficulty in naming quotients. Each set contains a special step. Any single column of these examples may be used to locate approximately the level where difficulty may be found.

	(1)	(2)	(3)	(4)	(5)
(*a*)	$20\overline{)40}$	$30\overline{)60}$	$40\overline{)80}$	$20\overline{)60}$	$40\overline{)40}$
(*b*)	$20\overline{)44}$	$30\overline{)36}$	$40\overline{)88}$	$10\overline{)65}$	$20\overline{)66}$
(*c*)	$21\overline{)42}$	$32\overline{)64}$	$43\overline{)86}$	$23\overline{)69}$	$41\overline{)82}$
(*d*)	$32\overline{)69}$	$21\overline{)89}$	$12\overline{)38}$	$31\overline{)96}$	$23\overline{)48}$
(*e*)	$20\overline{)54}$	$43\overline{)97}$	$21\overline{)70}$	$24\overline{)97}$	$23\overline{)59}$
(*f*)	$40\overline{)160}$	$80\overline{)480}$	$51\overline{)102}$	$61\overline{)122}$	$21\overline{)126}$
(*g*)	$35\overline{)105}$	$32\overline{)160}$	$34\overline{)108}$	$43\overline{)236}$	$52\overline{)171}$
(*h*)	$29\overline{)275}$	$58\overline{)187}$	$78\overline{)356}$	$29\overline{)222}$	$28\overline{)138}$
(*i*)	$63\overline{)428}$	$36\overline{)226}$	$55\overline{)411}$	$25\overline{)128}$	$35\overline{)251}$
(*j*)	$14\overline{)74}$	$15\overline{)90}$	$14\overline{)107}$	$18\overline{)165}$	$17\overline{)95}$

In sets (*a*) to (*g*) the trial quotient is the true quotient. In sets (*h*) to (*j*) the trial quotient is not the true quotient. In these sets the pupil faces the problem of first estimating the quotient and then checking his estimate by mental multiplication. He must repeat this checking until he has found the correct quotient figure. The pupil should be given considerable practice in naming quotients with specially prepared exercises, beginning with level (*a*) and proceeding to level (*j*).

6. Special Types of Difficulties in Long Division.

The chief new skills needed in long division, in addition to the ability to name quotients, are those concerned with the multiplication of the quotient figure involving carrying and the subtraction necessary in long division examples. Some authorities suggest that only one new difficulty be introduced at a time. Note the following examples:

(*a*)
$$\begin{array}{r} 22 \\ 31\overline{)682} \\ \underline{62} \\ 62 \\ \underline{62} \\ 0 \end{array}$$

In example (*a*) there is no carrying in multiplication and there is no borrowing in subtraction.

(*b*)
$$\begin{array}{r} 22 \\ 45\overline{)990} \\ \underline{90} \\ 90 \\ \underline{90} \\ 0 \end{array}$$

In example (*b*) there is carrying in multiplication, but no borrowing in subtraction.

(*c*)
$$\begin{array}{r} 23 \\ 22\overline{)506} \\ \underline{44} \\ 66 \\ \underline{66} \end{array}$$

In example (*c*) there is borrowing in subtracting 44 from 50, but there is no carrying in multiplication.

(*d*)
$$\begin{array}{r} 31 \\ 36\overline{)1116} \\ \underline{108} \\ 36 \\ \underline{36} \end{array}$$

In example (*d*) there are both carrying in multiplication and borrowing in subtraction.

Any one of the above types can be further complicated by introducing the factor of remainders. Two other factors that can increase the difficulty of these examples are the introduction of more difficult quotient figures and zero difficulties in quotients.

The following exercise contains the different types of examples in long division by means of which specific difficulties in long division can be located. Each example illustrates a new difficulty, either in process or in quotients.

1. $23\overline{)483}$	2. $25\overline{)575}$	3. $2\overline{)308}$
4. $31\overline{)345}$	5. $32\overline{)407}$	6. $35\overline{)428}$
7. $21\overline{)1155}$	8. $32\overline{)2592}$	9. $42\overline{)1008}$
10. $33\overline{)1786}$	11. $28\overline{)1896}$	12. $63\overline{)4285}$
13. $15\overline{)1065}$	14. $16\overline{)912}$	15. $35\overline{)10675}$
16. $72\overline{)14416}$	17. $26\overline{)52104}$	18. $37\overline{)74370}$

Appropriate remedial exercises must contain examples of the types given above. The examples should be given in a well graded series.

7. The Most Common Faults in Division.

In Table 5 are given the most common faults in long and short division, according to an analysis of data contained in the report of Buswell and John.[1]

[1] Buswell and John, *loc. cit.*

TABLE 5

Frequency of Occurrence of the Most Common Faults in Division of Whole Numbers

(Adapted from Buswell and John, page 139)

	Grades				Total
	III	IV	V	VI	
1. Errors in division combinations	35	55	59	42	191
2. Errors in subtraction	4	25	47	37	113
3. Errors in multiplication	1	20	48	36	105
4. Remainders:					
(a) Used remainder larger than divisor	1	17	39	29	86
(b) Neglected to use remainder within example	5	27	25	13	70
(c) Wrote remainders within example	8	11	17	13	49
(d) Omitted final remainder	4	16	18	11	49
(e) Used remainder without new dividend figure	0	6	14	9	29
(f) Wrote all remainders at the end of the example	1	0	6	2	9
(g) Added remainder to quotient	0	2	1	0	3
(h) Added remainder to next digit in dividend	0	0	0	1	1
Faulty procedures:					
(a) Found quotient by trial multiplication	1	8	49	24	82
(b) Omitted zero resulting from another digit	0	20	22	24	66
(c) Omitted digit in dividend	4	15	27	18	64
(d) Used short division form for long division	0	16	24	10	50
(e) Omitted zero resulting from zero in dividend	3	12	19	12	46
(f) Used long division form for short division	0	4	27	13	44
(g) Used too large a product	0	7	21	12	40
(h) Said example backwards	9	11	8	7	35
(i) Grouped many digits in dividend	1	4	12	5	22
(j) Used dividend or divisor as quotient	2	4	6	4	16
(k) Reversed dividend and divisor	8	3	2	2	15
(l) Used digits of divisor separately	0	1	8	1	10
(m) Used digits in dividend twice	0	2	5	2	9
(n) Used second digit in divisor to find quotient	0	1	5	1	7

TABLE 5 (*Continued*)

	Grades III	IV	V	VI	Total
(o) Began dividing from the right	1	1	4	1	7
(p) Split dividend	0	0	5	1	6
(q) Used endings to find quotient (long division)	0	2	3	1	6
(r) Added zeros to dividend when quotient not a whole number	0	0	1	2	3
(s) Wrote row of zeros	0	0	1	0	1
(t) Dropped zero in divisor, not in dividend	0	0	0	1	1
6. Counting:					
(a) Counted to get quotient	5	25	24	4	58
(b) Repeated part of multiplication table	4	15	27	9	55
(c) Counted in subtractions	3	15	18	6	42
(d) Found quotient by adding	1	3	6	4	14
7. Lapses, etc.:					
(a) Used wrong operation	17	17	24	6	64
(b) Derived unknown combination from known	1	6	11	8	26
(c) Had right answer but used wrong one	0	7	10	4	21
(d) Error in reading	3	2	10	2	17
(e) Misinterpreted table	0	2	5	2	9
8. Illegible figures	0	0	0	1	1
Total cases	44	77	103	76	300

As was found in each of the other processes, the chief faults in division are lack of knowledge of the fundamental combinations and counting to get the answer. These difficulties can be remedied by well organized practice. Many errors were made in subtraction, both in carrying in short division and in subtracting in long division examples. Many errors were also made in the multiplication of divisors and quotient figures. These data show that much of the deficiency in long division may be due to weakness in the other basic processes involved in the complex long division examples.

Another large group of errors is due to difficulty with remainders, mainly because of using remainders larger than the divisor, or because of neglect to use remainders within examples. Many faulty procedures are found, a large proportion of them involving zero difficulties and the method of finding the quotient.

PROBLEMS FOR STUDY, REPORTS, AND DISCUSSION

1. Do you think that the lists of skills in each process, as prepared by Miss Merton, are complete? Analyze them carefully. What changes, if any, would you make?

2. List the skills that are involved in solving the following examples:

$$\begin{array}{r} 78 \\ 49 \\ \underline{675} \end{array} \qquad \begin{array}{r} 7890 \\ \underline{-\ 1564} \end{array} \qquad \begin{array}{r} 786 \\ \underline{\times\ 35} \end{array} \qquad 65\overline{)6770}$$

3. Show how addition is used in subtraction, multiplication, and division.

4. What are the most common difficulties in addition? in subtraction? in multiplication? in division?

5. Construct a duplicate set of subtraction examples containing the same types of difficulties as those in the examples on page 131.

6. Show how subtraction is used in short division. Prepare a set of exercises that will give the pupils practice on the subtraction used in short division.

7. Make an analysis of the examples in naming quotients on page 153 and show the specific skills involved in each level of difficulty.

8. Examine textbooks to discover the methods pupils are taught to use in estimating quotients in long division.

9. Select some pupil whose work in arithmetic is inferior and by means of a standard diagnostic test make a diagnosis of his difficulties. Use the technique of psychological diagnosis described in the previous chapter. Select or prepare

a set of remedial exercises that you would use to overcome his difficulties.

10. Why might continued practice on exercises with which the pupil has difficulty result in failure to produce any improvement in his work?

11. Examine an arithmetic text and list the number of types of examples in a process that are presented in the new development units. Compare the number of steps in developments of processes in new and old texts.

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CHAPTER V

DIAGNOSIS OF DIFFICULTIES IN FRACTIONS

1. AN ANALYSIS OF THE SPECIFIC ABILITIES IN FRACTIONS.

The first step in the development of a diagnostic procedure in fractions is the analysis of the specific abilities involved in each of the processes. This is true because weakness in working fraction examples may be due to deficiency in one or more of these specific skills or abilities. Such an analysis of skills likewise facilitates a study of the instructional materials that are being used by the class to discover the possible shortcomings of their content. Diagnostic tests may be constructed on the basis of such an analysis. By means of such tests the teacher can locate the specific nature of the difficulty that the pupils have in any of the processes. Such an analysis of the specific skills in each process is also of great value to the textbook writer. This list, supplemented by a knowledge of the most difficult types of examples and of the major causes of pupil difficulty in each process, furnishes a valuable set of specifications to be used as the basis for the development of adequate instructional and practice units.

(a) Specific abilities involved in addition of fractions. The following list of items arrays the various new skills involved in the addition of fractions:

SKILL No.	DESCRIPTION OF SKILL
(1)	1. Knowledge that fractions must have a common denominator before they are to be added.

SKILL No. DESCRIPTION OF SKILL

(2) 2. Knowledge that to find the sum of two or more like fractions their numerators are to be added and their sum written over the common denominator.

3. Ability to reduce unlike fractions to a common denominator:

(3) (*a*) When the common denominator is one of the given denominators: $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$; $4\frac{1}{3} + \frac{1}{6} = 4\frac{1}{2}$.

(4) (*b*) When a common denominator is the product of given denominators: $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$; $2\frac{1}{4} + 3\frac{1}{3} = 5\frac{7}{12}$.

4. Knowledge of procedure to use in:

(5) (*a*) Adding like fractions.

(6) (*b*) Adding unlike fractions.

(7) (*c*) Adding whole numbers and fractions.

(8) (*d*) Adding whole and mixed numbers.

(9) (*e*) Adding fractions and mixed number (all types of fractions).

(10) (*f*) Adding two or more mixed numbers (all types of fractions).

5. Ability to reduce sums to simplest form:

(11) (*a*) Ability to recognize answer already in simplest form.

(12) (*b*) Ability to reduce proper fractions: $\frac{4}{6} = \frac{2}{3}$; $\frac{8}{12} = \frac{2}{3}$.

(*c*) Ability to reduce improper fractions:

(13) (1) Simple reduction, answer a mixed number: $\frac{11}{5} = 2\frac{1}{5}$.

(14) (2) Simple reduction, answer a whole number: $\frac{8}{4} = 2$; $\frac{2}{2} = 1$.

(15) (3) Further reduction, answer a mixed number: $\frac{12}{8} = 1\frac{4}{8} = 1\frac{1}{2}$; $\frac{18}{4} = 4\frac{2}{4} = 4\frac{1}{2}$.

(*d*) Ability to reduce mixed numbers:

(16) (1) No reduction required: $9\frac{1}{3} + 7\frac{1}{4} = 16\frac{7}{12}$.

NOTE.—An analysis of descriptive elements to be used in analyzing practice in fractions, differing in make-up from the lists here given, has been published by F. Knight, E. Luse, and G. Ruch, *Problems in the Teaching of Arithmetic* (Iowa City, Iowa: Iowa Supply Store, 1925).

(17) (2) Simple reduction: $12\frac{4}{8} = 12\frac{1}{2}$; $16\frac{6}{12} = 16\frac{1}{2}$.

(18) (3) Carrying, answer a whole number: $12\frac{8}{4} = 14$.

(19) (4) Carrying, simple reduction, answer a mixed number: $7\frac{9}{2} = 11\frac{1}{2}$.

(20) (5) Carrying, further reduction, answer a mixed number: $7\frac{8}{6} = 8\frac{2}{6} = 8\frac{1}{3}$.

How to use the list of skills in analyzing an example: This list of items may be used in making an analysis of the skills involved in working such an example as the following:

$$\begin{array}{l} 7\frac{1}{2} = 7\frac{2}{4} \\ \underline{8\frac{3}{4}} = \underline{8\frac{3}{4}} \\ \qquad 15\frac{5}{4} = 15 + 1\frac{1}{4} = 16\frac{1}{4} \end{array}$$

The skills involved in this example are skills 1, 2, 3, 10, 13, 19, described in the list above.

The skills involved in working the following examples differ from those listed for the above example, as can be determined by a comparison of the skills indicated.

(1)	$\begin{array}{r} 7\frac{3}{4} \\ \underline{\frac{3}{4}} \\ 8\frac{1}{2} \end{array}$	Skills required are Nos. 1, 2, 9, 15, 20.	(2)	$\begin{array}{r} 8\frac{1}{3} \\ \underline{6\frac{1}{4}} \\ 14\frac{7}{12} \end{array}$	Skills required are Nos. 1, 2, 4, 10, 16.

In a similar way an analysis can be made of the skills involved in any example in the addition of fractions. The diagnostic test in addition of fractions on page 164 contains a large variety of types of examples, each of which contains a different combination of the unit skills listed above. The reader can readily make an analysis of the unit skills involved in each example on the basis of the list of skills given above. Such analysis will make clear the complexity of the various types of examples in the process.

BRUECKNER TEST IN FRACTIONS.

Date.................................... School ....

Name........................Grade....................Age................Sex...........

Find the answers to the following addition examples:

	a	*b*	*c*	*d*	*e*
1.	$\frac{1}{3}$ $\frac{1}{3}$	$\frac{1}{8}$ $\frac{5}{8}$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{4}{5}$ $\frac{2}{5}$	$\frac{3}{4}$ $\frac{3}{4}$
2.	5 $\frac{1}{4}$	$\frac{2}{3}$ 7	4 $1\frac{2}{3}$	$3\frac{1}{2}$ 8	$2\frac{1}{4}$ $\frac{1}{4}$ 4
3.	$3\frac{1}{5}$ $\frac{1}{5}$	$\frac{1}{4}$ $2\frac{1}{4}$	$2\frac{1}{3}$ $7\frac{2}{3}$	$3\frac{3}{5}$ $2\frac{4}{5}$	$5\frac{5}{6}$ $2\frac{5}{6}$
4.	$\frac{1}{2}$ $\frac{1}{4}$	$\frac{1}{2}$ $\frac{1}{6}$	$\frac{1}{4}$ $\frac{1}{2}$ $\frac{1}{6}$	$\frac{1}{3}$ $\frac{5}{6}$	$\frac{7}{12}$ $\frac{3}{4}$
5.	$3\frac{2}{3}$ $2\frac{1}{6}$	$1\frac{1}{4}$ $3\frac{7}{12}$	$1\frac{1}{4}$ $\frac{1}{4}$ $2\frac{1}{2}$	$7\frac{3}{4}$ $3\frac{1}{2}$	$2\frac{2}{3}$ $7\frac{5}{6}$
6.	$\frac{1}{4}$ $\frac{1}{3}$	$\frac{1}{3}$ $\frac{1}{5}$ $\frac{3}{10}$	$\frac{3}{4}$ $\frac{1}{6}$ $\frac{1}{8}$	$\frac{3}{5}$ $\frac{1}{2}$ $\frac{1}{4}$	$\frac{2}{3}$ $\frac{4}{5}$ $\frac{7}{10}$
7.	$4\frac{1}{5}$ $1\frac{1}{4}$	$1\frac{1}{6}$ $7\frac{3}{5}$ $3\frac{1}{6}$	$\frac{5}{6}$ $\frac{3}{4}$ $\frac{5}{12}$	$6\frac{2}{3}$ $4\frac{3}{4}$	$5\frac{3}{8}$ $7\frac{1}{4}$ $\frac{1}{4}$
8.	$5\frac{1}{12}$ $\frac{5}{6}$	$\frac{5}{12}$ $1\frac{1}{3}$	$\frac{2}{3}$ $\frac{1}{4}$ $4\frac{1}{12}$	$\frac{2}{5}$ $1\frac{2}{3}$	$3\frac{17}{24}$ $4\frac{5}{6}$ $3\frac{5}{8}$

(b) **Specific abilities involved in subtraction of fractions.** The following is a list of the specific skills involved in the subtraction of fractions:

SKILL NO. — DESCRIPTION OF SKILL

(1) 1. Knowledge that fractions must have a common denominator before they can be subtracted.

(2) 2. Knowledge that numerators of like fractions are to be subtracted and the difference written over the common denominator.

3. Ability to reduce unlike fractions to a common denominator:

(3) (*a*) When common denominator is one of the given denominators: $\frac{1}{2} - \frac{1}{4} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$; $4\frac{1}{2} - 3\frac{1}{4} = 4\frac{2}{4} - 3\frac{1}{4} = 1\frac{1}{4}$.

(4) (*b*) When common denominator is the product of given denominators: $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$; $3\frac{1}{3} - \frac{1}{8} = 3\frac{5}{24}$.

4. Ability to use procedure in subtracting fractions and mixed numbers, not involving borrowing, in—

(5) (*a*) Subtracting like fractions: $\frac{8}{8} - \frac{3}{8} = \frac{5}{8}$; $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

(6) (*b*) Subtracting unlike fractions: $\frac{7}{8} - \frac{1}{6} = \frac{17}{24}$.

(7) (*c*) Subtracting fractions from mixed numbers, no zero difficulty: $3\frac{1}{4} - \frac{1}{8} = 3\frac{1}{8}$.

(8) (*d*) Subtracting two mixed numbers, no zero difficulty: $5\frac{1}{2} - 2\frac{1}{4} = 3\frac{1}{4}$.

(*e*) Special types of zero difficulties with—

(9) (1) Answer a proper fraction, zero not expressed: $7\frac{1}{2} - 7\frac{1}{4} = \frac{1}{4}$.

(10) (2) Answer a whole number, zero not expressed: $7\frac{1}{8} - 3\frac{1}{8} = 4$; $4\frac{2}{3} - \frac{2}{3} = 4$.

(11) (3) Answer is zero, and zero is expressed: $\frac{1}{8} - \frac{1}{8} = 0$; $7\frac{1}{8} - 7\frac{1}{8} = 0$.

5. Ability to subtract a whole number from a mixed number:

(12) (*a*) Knowledge that a fraction less zero gives the original fraction.

(13) (*b*) No zero difficulty: $4\frac{1}{2} - 2 = 2\frac{1}{2}$.

SKILL NO. DESCRIPTION OF SKILL

(14) (*c*) Zero difficulty, zero not expressed: $4\frac{1}{2} - 4 = \frac{1}{2}$.

6. Ability to use procedure in subtracting fractions and mixed numbers from whole numbers, which involves borrowing—

(15) (*a*) When whole number is unit (1): $1 - \frac{3}{4} = \frac{4}{4} - \frac{3}{4} = \frac{1}{4}$.

(16) (*b*) When whole number is greater than one, no zero difficulty: $7 - \frac{3}{4} = 6\frac{4}{4} - \frac{3}{4} = 6\frac{1}{4}$; $8 - 4\frac{3}{4} = 7\frac{4}{4} - 4\frac{1}{4} = 3\frac{3}{4}$.

(17) (*c*) When whole number is greater than one, zero difficulty, zero not expressed: $8 - 7\frac{1}{2} = 7\frac{2}{2} - 7\frac{1}{2} = \frac{1}{2}$.

7. Ability to use procedure in subtracting fractions and mixed numbers from mixed numbers, which involves borrowing:

(18) (*a*) Ability to change the form of the minuend in examples containing like fractions: $7\frac{1}{4} - \frac{3}{4} = 6\frac{5}{4} - \frac{3}{4} = 6\frac{1}{2}$; $7\frac{1}{8} - 2\frac{3}{8} = 6\frac{9}{8} - 2\frac{3}{8} = 4\frac{3}{4}$.

(19) (*b*) Ability to change the form of the minuend in examples containing unlike fractions: $6\frac{1}{2} - \frac{3}{4} = 6\frac{2}{4} - \frac{3}{4} = 5\frac{6}{4} - \frac{3}{4} = 5\frac{3}{4}$.

NOTE.—After the form of the minuend in 6 and 7 is changed, the skills listed under 4 become operative. No new skills are required.

8. Ability to reduce answers to simplest form:

(20) (*a*) Ability to recognize answers already in simplest form.

(21) (*b*) Ability to reduce proper fractions to lowest terms: $\frac{2}{4} = \frac{1}{2}$.

(22) (*c*) Ability to reduce mixed numbers to simplest form: $4\frac{2}{4} = 4\frac{1}{2}$.

(23) (*d*) Knowledge that fraction with 0 numerator equals zero: $\frac{1}{8} - \frac{1}{8} = \frac{0}{8} = 0$.

This list of skills may be used in making an analysis of the skills involved in any example in subtracting fractions. The following examples illustrate the method:

$$(1)\quad \begin{array}{r} 7\tfrac{1}{2} = 7\tfrac{2}{4} \\ \underline{-\ \tfrac{1}{4}} = \underline{\ \tfrac{1}{4}} \\ 7\tfrac{1}{4} \end{array} \quad \text{Skills required are Nos. 1, 2, 3, 7, 20.}$$

$$(2)\quad \begin{array}{r} 8\tfrac{1}{6} = 8\tfrac{1}{6} = 7\tfrac{7}{6} \\ \underline{-7\tfrac{2}{3}} = 7\tfrac{4}{6} = \underline{7\tfrac{4}{6}} \\ \tfrac{3}{6} = \tfrac{1}{2} \end{array} \quad \text{Skills required are Nos. 1, 2, 3, 19, 9, 21.}$$

Similar analyses may be made of the unit skills involved in the solution of the examples in the diagnostic test in fractions on page 168.

(c) **Specific abilities involved in multiplication of fractions.** The following list of items arrays the unit skills involved in multiplication of fractions:

SKILL NO. DESCRIPTION OF SKILL

(1) 1. Knowledge that fractions need not be reduced to a common denominator before multiplying.

(2) 2. Knowledge that "of" means "times," as in $\frac{1}{2}$ of 4.

(3) 3. Knowledge that the product of the numerators gives the numerator of the answer and that the product of the denominators gives the denominator of the answer.

(4) 4. Ability to reduce mixed numbers to improper fractions: $7\frac{1}{2} = \frac{15}{2}$.

(5) 5. Knowledge that a whole number, as 4, may be expressed with a denominator, as $\frac{4}{1}$.

6. Ability to apply procedure to use in—

(6) (*a*) Multiplying proper fractions, with unit numerators: $\frac{1}{4} \times \frac{1}{8} = \frac{1}{32}$.

(7) (*b*) Multiplying proper fractions with greater than unit numerators: $\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$.

(8) (*c*) Multiplying proper fractions and whole numbers: $\frac{1}{4} \times 4 = 1$.

(9) (*d*) Multiplying proper fractions and mixed numbers: $\frac{2}{3} \times 1\frac{1}{4} = \frac{5}{6}$.

(10) (*e*) Multiplying whole and mixed numbers: $4 \times 7\frac{7}{8} = 31\frac{1}{2}$.

Brueckner Test in Fractions:

Date.................................... School....................................

Name.......................Grade.......................Age............Sex...........

Find the answers to the following subtraction examples:

	a	*b*	*c*	*d*	*e*	*f*	*g*
1.	$\frac{5}{4}$ $\frac{3}{4}$	$\frac{8}{8}$ $\frac{3}{8}$	$\frac{2}{3}$ $\frac{1}{3}$	$\frac{5}{8}$ $\frac{1}{8}$	$4\frac{2}{3}$ $\frac{1}{3}$	$4\frac{2}{3}$ $\frac{2}{3}$	$7\frac{5}{6}$ $\frac{1}{6}$
2.	$5\frac{1}{4}$ 3	$4\frac{3}{5}$ $2\frac{1}{5}$	$3\frac{3}{4}$ $2\frac{3}{4}$	$7\frac{7}{8}$ $2\frac{1}{8}$	$1\frac{5}{8}$ $1\frac{1}{8}$	9 $\frac{1}{2}$	5 $1\frac{1}{3}$
3.	3 $2\frac{5}{12}$	$6\frac{1}{3}$ $\frac{2}{3}$	$7\frac{1}{8}$ $\frac{5}{8}$	$10\frac{1}{3}$ $4\frac{2}{3}$	$9\frac{3}{5}$ $8\frac{4}{5}$	$4\frac{1}{8}$ $1\frac{7}{8}$	$\frac{1}{2}$ $\frac{1}{4}$
4.	$\frac{1}{2}$ $\frac{1}{6}$	$1\frac{1}{3}$ $\frac{2}{3}$	$3\frac{3}{4}$ $\frac{1}{2}$	$4\frac{5}{6}$ $\frac{1}{3}$	$3\frac{3}{4}$ $1\frac{1}{2}$	$7\frac{3}{5}$ $1\frac{1}{10}$	$4\frac{1}{4}$ $\frac{5}{8}$
5.	$7\frac{1}{2}$ $2\frac{3}{4}$	$3\frac{1}{6}$ $1\frac{2}{3}$	$\frac{1}{2}$ $\frac{1}{2}$	1 $\frac{1}{2}$	$2\frac{1}{2}$ $2\frac{1}{2}$	$2\frac{3}{4}$ $1\frac{1}{8}$	$1\frac{1}{10}$ $\frac{4}{5}$
6.	$1\frac{1}{2}$ $\frac{5}{6}$	$1\frac{5}{6}$ $1\frac{1}{2}$	$3\frac{1}{4}$ $2\frac{5}{8}$	$2\frac{1}{6}$ $\frac{2}{3}$	$3\frac{1}{4}$ $1\frac{2}{3}$	$6\frac{1}{2}$ $6\frac{2}{5}$	$\frac{1}{3}$ $\frac{1}{4}$
7.	$\frac{9}{16}$ $\frac{1}{2}$	$2\frac{3}{8}$ $\frac{1}{3}$	$8\frac{1}{2}$ $\frac{2}{3}$	$4\frac{5}{6}$ $1\frac{2}{5}$	$14\frac{3}{4}$ $14\frac{1}{2}$	$1\frac{1}{8}$ $\frac{5}{8}$	$6\frac{1}{4}$ $5\frac{2}{3}$
8.	$1\frac{1}{3}$ $\frac{1}{2}$	$10\frac{1}{2}$ 10	$2\frac{1}{4}$ $1\frac{3}{4}$	$9\frac{1}{10}$ $8\frac{1}{2}$			

(11) (*f*) Multiplying mixed numbers: $4\frac{1}{2} \times 4\frac{2}{3} = 21$.

7. Ability to express products in simplest form (if cancellation is not used), when product is expressed as—

(*a*) Proper fraction.

(12) (1) Reducible: $\frac{3}{6} = \frac{1}{2}$.

(13) (2) Irreducible: $\frac{2}{3}$.

(*b*) Improper fraction, which gives—

(14) (1) Product a whole number: $\frac{12}{4} = 3$; $\frac{4}{4} = 1$.

(15) (2) Product a mixed number, simple reduction: $\frac{12}{5} = 2\frac{2}{5}$.

(16) (3) Product a mixed number, involving further reduction: $\frac{18}{8} = 2\frac{2}{8} = 2\frac{1}{4}$.

8. Ability to cancel, which involves—

(17) (*a*) Ability to determine when cancellation is not possible.

(18) (*b*) Ability to recognize common factors.

(*c*) Ability to indicate cancellation in numerator and denominator.

(19) (1) Single cancellation: $\frac{2}{\cancel{3}} \times \frac{\cancel{3}}{5} = \frac{2}{5}$.

(20) (2) Double cancellation: $\dfrac{\overset{1}{\cancel{3}}}{\underset{4}{\cancel{8}}} \times \dfrac{\overset{5}{\cancel{10}}}{\underset{7}{\cancel{21}}} = \dfrac{5}{28}$

(21) (*d*) Ability to manipulate unit terms remaining after complete cancellation:

$$\frac{\overset{1}{\cancel{4}}}{\cancel{5}} \times \frac{\cancel{5}}{\underset{2}{\cancel{8}}} = \frac{1}{2};\quad \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{4}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{3}}} = \frac{1}{1} = 1.$$

NOTE.—Cancellation is merely a short-cut method of reduction before the respective multiplications have been made. It should probably not be taught until the basic procedure has been firmly established. Otherwise interference with important habits may result in serious difficulty.

The skills involved in solving typical examples in the multiplication of fractions are analyzed below:

(1) $7\frac{1}{4} \times 2\frac{2}{3} = \dfrac{29}{\underset{1}{\cancel{4}}} \times \dfrac{\overset{2}{\cancel{8}}}{3} = \dfrac{58}{3} = 19\frac{1}{3}$. Skills required are Nos. 1, 3, 4, 11, 15, 18, 19.

(2) $\frac{4}{5} \times 8 = \frac{32}{5} = 6\frac{2}{5}$. Skills involved are 1, 3, 5, 15, 17.

The diagnostic test in multiplication of fractions on this page contains a wide variety of types of examples in which these unit skills occur in varying combinations.

Brueckner Test in Fractions

Date.................................... School....................................

Name............................ Grade.................... Age................ Sex..............

Find the answers to the following multiplication examples:

	a	b	c	d	e
1.	$\frac{1}{4} \times 8 =$	$\frac{1}{6} \times 3 =$	$\frac{2}{3} \times 2 =$	$\frac{2}{3} \times 16 =$	$\frac{5}{6} \times 14 =$
2.	$2 \times \frac{4}{5} =$	$3 \times \frac{1}{3} =$	$4 \times \frac{1}{2} =$	$12 \times \frac{1}{5} =$	$8 \times \frac{1}{6} =$
3.	$\frac{1}{4} \times \frac{1}{6} =$	$\frac{1}{4} \times \frac{0}{6} =$	$\frac{1}{2} \times \frac{7}{10} =$	$\frac{5}{6} \times \frac{3}{7} =$	$\frac{5}{8} \times \frac{15}{25} \times \frac{1}{2} =$
4.	$6 \times 2\frac{1}{3} =$	$7 \times 4\frac{1}{2} =$	$6 \times 3\frac{3}{4} =$	$2 \times 4\frac{1}{2} =$	$\frac{2}{3} \times \frac{2}{3} \times 4\frac{3}{4} =$
5.	$2\frac{1}{4} \times \frac{1}{4} =$	$3\frac{1}{5} \times 2 =$	$3\frac{1}{6} \times 4 =$	$3\frac{2}{3} \times 9 =$	$\frac{4}{5} \times 10 \times \frac{3}{8} =$
6.	$\frac{1}{4} \times 2\frac{1}{2} =$	$\frac{1}{2} \times 1\frac{1}{5} =$	$\frac{5}{6} \times 6\frac{2}{5} =$	$\frac{2}{5} \times 3\frac{2}{3} =$	$\frac{3}{8} \times 6 \times 2\frac{1}{3} =$
7.	$3\frac{1}{3} \times \frac{3}{8} =$	$3\frac{1}{3} \times \frac{1}{8} =$	$7\frac{1}{2} \times \frac{2}{5} =$	$6\frac{4}{5} \times \frac{3}{7} =$	$4 \times 3\frac{3}{5} \times 7 =$
8.	$2\frac{1}{2} \times 2\frac{1}{4} =$	$6\frac{1}{4} \times 6\frac{2}{5} =$	$3\frac{3}{4} \times 3\frac{1}{3} =$	$3\frac{2}{7} \times 2\frac{2}{3} =$	$4\frac{3}{8} \times 2\frac{6}{7} \times 1\frac{4}{5} =$
9.	$24 \times 3\frac{1}{8}$	$36\frac{5}{6} \times 12$	$18 \times 15\frac{3}{4}$	$25\frac{1}{2} \times 27$	$23 \times 18\frac{3}{5}$

(d) **Specific abilities involved in division of fractions.** The following list of items arrays the skills involved in division of fractions. It is assumed that the method of inversion is used.

Skill No.	Description of Skills
(1)	1. The knowledge that the divisor must be inverted as the important new step in a division example.
(2)	2. Knowledge that a whole number such as 4 may be thought of as having the denominator 1, thus $\frac{4}{1}$; $\frac{2}{3} \div 4 = \frac{2}{3} \div \frac{4}{1} = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$; $4 \div \frac{1}{3} = \frac{4}{1} \times \frac{3}{1} = 12$.
(3)	3. Ability to reduce a mixed number to an improper fraction: $4\frac{1}{2} = \frac{9}{2}$.
(4)	4. Knowledge that mixed numbers should almost always first be reduced to improper fractions: $2\frac{1}{2} \div 1\frac{1}{4} = \frac{5}{2} \div \frac{5}{4} = 2$.
(5)	5. Knowledge that after the divisor is inverted the procedure is the same as in an example in multiplication of fractions, thus involving many of the specific skills and knowledges listed for that process.
	6. The ability to invert the divisor when it is—
(6)	(*a*) A proper fraction: $\div \frac{1}{2} = \times \frac{2}{1}$; $\div \frac{4}{5} = \times \frac{5}{4}$.
(7)	(*b*) An integer: $\div 4 = \times \frac{1}{4}$.
(8)	(*c*) A mixed number: $\div 2\frac{1}{2} = \div \frac{5}{2} = \times \frac{2}{5}$.
	7. After the divisor is inverted, the example involves the ability to work an example in which—
(9)	(*a*) An integer is multiplied by a proper or an improper fraction: $5 \div \frac{2}{3} = 5 \times \frac{3}{2}$; $5 \div 2\frac{1}{2} = 5 \div \frac{5}{2} = 5 \times \frac{2}{5}$.
(10)	(*b*) A proper or an improper fraction is multiplied by a proper or an improper fraction: $\frac{2}{3} \div \frac{3}{5} = \frac{2}{3} \times \frac{5}{3}$; $2\frac{1}{2} \div 4 = \frac{5}{2} \times \frac{1}{4}$; $2\frac{1}{2} \div \frac{1}{2} = \frac{5}{2} \times \frac{2}{1}$.
(11)	8. Ability to cancel when the form has been changed to a multiplication example. See skills listed under multiplication.
	9. Ability to reduce the answer to the simplest form involving—
(12)	(*a*) Ability to recognize fractions already reduced to lowest terms.
(13)	(*b*) Ability to reduce proper fractions to lowest terms.

(*c*) Ability to reduce improper fractions to simplest form:

(14) (1) As a whole number: $\frac{4}{2} = 2$.

(15) (2) As a mixed number, no second reduction: $\frac{9}{8} = 1\frac{1}{8}$.

(16) (3) As a mixed number, involving further reduction: $\frac{10}{8} = 1\frac{2}{8} = 1\frac{1}{4}$.

The following illustrate the analysis of skills involved in three typical examples in division of fractions.

(1) $5 \div 2\frac{1}{2} =$ Skills in division, 1, 2, 3, 4, 5, 8, 9, 11, 14.

$$\frac{\overset{1}{\cancel{5}}}{1} \times \frac{2}{\underset{1}{\cancel{5}}} = \frac{2}{1} = 2$$

(2) $5 \div 7\frac{1}{2} =$ Skills in division, 1, 2, 3, 4, 5, 8, 9, 11, 12.

$$\frac{\overset{1}{\cancel{5}}}{1} \times \frac{2}{\underset{3}{\cancel{15}}} = \frac{2}{3}$$

(3) $5 \div 3\frac{3}{4} =$ Skills in division, 1, 2, 3, 4, 5, 8, 9, 11, 15.

$$\frac{\overset{1}{\cancel{5}}}{1} \times \frac{4}{\underset{3}{\cancel{15}}} = \frac{4}{3} = 1\frac{1}{3}$$

Division examples 1, 2, and 3, each involve the division of a whole number by a mixed number; yet each example contains a specific element which makes it differ from the other two in an important way. Division skills 1, 2, 3, 4, 5, 8, 9, and 11 are common to the three examples. The difference between the types is revealed by the remaining element which varies in each example, namely, the nature of the answer. In example 1 the answer is an improper fraction which reduces to a whole number,

skill 14; in example 2 the answer is a proper fraction, skill 12; in example 3 the answer is a mixed number, skill 15. Similar analyses may be made of the skills involved in working the division examples found in the diagnostic test on this page.

The examples contained in the diagnostic tests in fractions are the important types that should be included in the practice materials. They contain, in various combinations, all of the skills listed under specific abilities on pages 171 to 172. Unless all of these types are present there is a possibility that the pupil may not be able to work examples similar to missing types encountered in life outside the school.

Brueckner Test in Fractions

Date.................................... School....................................

Name........................Grade..........................Age.............Sex.............

Find the answers to the following division examples:

	a	*b*	*c*	*d*	*e*
1.	$5 \div \frac{1}{2} =$	$5 \div \frac{2}{3} =$	$12 \div \frac{3}{4} =$	$14 \div \frac{4}{5} =$	$2 \div \frac{4}{5} =$
2.	$\frac{1}{7} \div \frac{1}{6} =$	$\frac{1}{9} \div \frac{1}{3} =$	$\frac{3}{8} \div \frac{3}{8} =$	$\frac{10}{15} \div \frac{1}{2} =$	$\frac{5}{6} \div \frac{3}{10} =$
3.	$\frac{1}{2} \div \frac{1}{4} =$	$1\frac{1}{4} \div \frac{1}{3} =$	$1\frac{1}{2} \div \frac{1}{2} =$	$1\frac{3}{8} \div \frac{3}{10} =$	$9 \div 8 =$
4.	$\frac{1}{5} \div 5 =$	$\frac{5}{6} \div 4 =$	$\frac{5}{6} \div 5 =$	$\frac{4}{5} \div 6 =$	$10 \div 6 =$
5.	$1\frac{1}{3} \div 5 =$	$1\frac{1}{5} \div 4 =$	$5\frac{2}{3} \div 5 =$	$5\frac{3}{5} \div 2 =$	$2 \div 14 =$
6.	$\frac{1}{3} \div 1\frac{1}{2} =$	$\frac{3}{8} \div 1\frac{2}{3} =$	$\frac{5}{12} \div 1\frac{5}{6} =$	$\frac{1}{6} \div 1\frac{1}{2} =$	$12 \div 6\frac{3}{5} =$
7.	$2 \div 2\frac{2}{3} =$	$3 \div 4\frac{1}{2} =$	$5 \div 2\frac{3}{4} =$	$5 \div 2\frac{1}{2} =$	$4\frac{1}{2} \div 2\frac{1}{4} =$
8.	$1\frac{1}{5} \div 3\frac{1}{2} =$	$1\frac{1}{3} \div 3\frac{1}{3} =$	$2\frac{3}{8} \div 2\frac{3}{8} =$	$3\frac{1}{3} \div 1\frac{3}{4} =$	$3\frac{3}{8} \div 1\frac{1}{4} =$

These tests are not standardized for rate of work nor are there norms or standards of accuracy. Their func-

tion is not to measure ability but to make it possible for the teacher to inventory the complete area of each process and to locate the types of examples which may be presenting difficulty for the class or for an individual pupil. This is vital at the time the processes in fractions are first being taught.

The diagnostic test in reduction of fractions given below supplies material by means of which the teacher can determine the specific element that may be causing difficulty in reduction or in changing the form of fractions. It will be shown that it is this phase of the work that is a major source of errors in all processes.

It is of course obvious that the ability of the pupil to work all examples that fit the description of a particular type is not measured by his ability to work a single example of the type. However, the fact that he can work a single example of the type means that he knows the procedure to use in working similar examples. To measure the ability of the pupil to work examples of a given type, a test containing from fifteen to twenty such examples should be given. At present no such tests are available.

Diagnostic Test in Reduction of Fractions[1]

1. Reduce the following fractions to lowest terms:

$\frac{9}{12} = \qquad \frac{15}{18} = \qquad \frac{6}{15} = \qquad \frac{6}{8} = \qquad \frac{10}{14} =$

2. Supply the missing terms in each of the following fractions:

$\frac{1}{2} = \frac{}{4} = \frac{}{8} = \frac{}{16} = \frac{}{10} = \frac{}{14}$

$\frac{1}{4} = \frac{}{16} \qquad \frac{1}{8} = \frac{}{16} \qquad \frac{4}{5} = \frac{}{10} \qquad \frac{2}{3} = \frac{}{9} \qquad \frac{3}{4} = \frac{}{8}$

3. Change the following improper fractions to mixed numbers. Express all answers in lowest terms.

$\frac{9}{6} = \qquad \frac{12}{8} = \qquad \frac{18}{12} = \qquad \frac{20}{16} = \qquad \frac{11}{3} = \qquad \frac{14}{6} =$

---

[1] Triangle Arithmetics, Book II, Part II, Page 217.

4. Change the following mixed numbers to improper fractions:

$1\frac{1}{4} =$ $2\frac{1}{2} =$ $7\frac{1}{3} =$ $5\frac{4}{5} =$ $8\frac{7}{8} =$ $9\frac{3}{11} =$

5. Change these pairs of unlike fractions to fractions with a lowest common denominator. The first example is already worked correctly.

a	*b*	*c*	*d*	*e*	*f*
$\frac{1}{4} = \frac{1}{4}$	$\frac{7}{8} =$	$\frac{1}{3} =$	$\frac{7}{8} =$	$\frac{3}{4} =$	$\frac{5}{6} =$
$\frac{1}{2} = \frac{2}{4}$	$\frac{3}{4} =$	$\frac{4}{5} =$	$\frac{1}{6} =$	$\frac{2}{3} =$	$\frac{3}{4} =$

6. Change $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ to twelfths.

7. Change $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{7}{8}$ to fractions having a lowest common denominator.

8. Supply the missing numerators in the following:

$7\frac{1}{2} = 6\frac{}{2}$	$5\frac{2}{3} = 4\frac{}{3}$	$6 = 5\frac{}{4}$	$8 = 7\frac{}{5}$
$3\frac{1}{2} = 2\frac{}{4}$	$5\frac{2}{3} = 4\frac{}{6}$	$9\frac{7}{8} = 8\frac{}{24}$	$15\frac{1}{6} = 14\frac{}{18}$

9. Reduce the following to simplest form:

$3\frac{9}{6} =$ $8\frac{4}{4} =$ $12\frac{12}{11} =$ $8\frac{8}{2} =$

10. Express the following as fractional parts of 100:

$12\frac{1}{2}$,	25,	75,	50,	$33\frac{1}{3}$,	$87\frac{1}{2}$
$66\frac{2}{3}$,	$37\frac{1}{2}$,	40,	$83\frac{1}{3}$,	60,	$62\frac{1}{2}$

(e) **Basis of Analysis of Fractions Proposed by Osburn.** Osburn[1] has proposed a somewhat different technique than the method just described for analyzing the procedures used in solving fraction examples. This technique makes very clear the complexity in the number and sequence of steps involved. Failure of the pupil to take any of the steps in the correct order will result in confusion and error. The following example illustrates Osburn's suggested procedure:

[1] W. S. Osburn, *Corrective Arithmetic*, Vol. 2 (Boston: Houghton Mifflin Company, 1929), pp. 39–46.

$$
\begin{array}{rl}
4\tfrac{1}{2} = & 4\tfrac{3}{6} \\
2\tfrac{2}{3} = & 2\tfrac{4}{6} \\
\hline
 & 6\tfrac{7}{6} = 6 + 1\tfrac{1}{6} = 7\tfrac{1}{6}
\end{array}
$$

The Steps are:

(1) Multiply 2 and 3...... M
(2) Divide 6 by 2......... D
(3) Multiply 1 by 3....... M
(4) Divide 6 by 3......... D
(5) Multiply 2 by 2....... M
(6) Add 3 and 4.......... A
(7) Add 4 and 2.......... A
(8) Divide 7 by 6......... D
(9) Subtract 6 from 7..... S
(10) Add $1\frac{1}{6}$ and 6......... A

Summary: M D M D M A A D S A

M stands for multiplication, S for subtraction, A for addition, D for division, I for invert.

In solving this example there are ten steps. Should one be surprised if the pupils find such examples difficult?

Similar analyses may be made for solutions of examples in other processes. The following analyses illustrate the complexity of steps involved in solving examples in the other processes in fractions:

Subtraction.

$3\frac{1}{3} - 2\frac{1}{2} =$ M D M D M A A S S (9 steps) or
M D M D M S A S S (9 steps)

Multiplication.

$3\frac{1}{5} \times \frac{3}{8} =$ M A D D M M D (7 steps)
$7\frac{2}{9} \times 2\frac{1}{10} =$ M A M A D D D D M M D S (12 steps)

Division.

$15 \div 1\frac{3}{4} =$ M A I M M D S (7 steps)
$7\frac{1}{2} \div 5\frac{1}{3} =$ M A M A I M M D M S (10 steps)

2. Relative Difficulty of Types of Examples in Fractions.

It is important that provision be made for practice on each of the types of examples in fractions. This is made clear by available data as to the relative difficulty

of types of examples in each process. To determine the difficulty of the types of examples found in the diagnostic tests previously described Kelly[1] gave the tests to approximately 800 pupils in Minnesota and Wisconsin schools at the end of the first semester in the sixth grade. These pupils had completed all work in fractions and in addition had a careful review of fractions.

Table 6 contains the results of this investigation. The table contains an analysis of the skills involved in each of the examples in the four tests. It also gives the per cent of the pupils who solved each example incorrectly; and the rank of difficulty of each example, based on these per cents of error.

TABLE 6

Analysis of the Examples in the Diagnostic Tests in Fractions as to Skills and Difficulty

A. ADDITION (40 Types)

Group I. Addition of Proper Fractions having Common Denominators

Type	Example	Description	Per Cent of Error	Rank of Difficulty
1	$\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$	Similar denominators. Sum is less than 1. No reduction.	5.0	1
2	$\frac{1}{8}+\frac{5}{8}=\frac{6}{8}=\frac{3}{4}$	Similar denominators. Sum is less than 1. Reduction.	14.9	14
3	$\frac{1}{2}+\frac{1}{2}=1$	Similar denominators. Sum is 1.	8.0	4
4	$\frac{4}{5}+\frac{2}{5}=\frac{6}{5}=1\frac{1}{5}$	Similar denominators. Numerators more than 1. Sum is more than 1. No reduction.	10.0	8
5	$\frac{3}{4}+\frac{3}{4}=\frac{6}{4}=1\frac{1}{2}$	Similar denominators. Numerators more than 1. Sum is more than 1. Reduction.	10.1	18

[1] F. Kelly, "An Analysis of the Relative Difficulty of Types of Examples in Fractions," Master's Thesis, Unpublished (Minneapolis, Minnesota: University of Minnesota, 1929).

TABLE 6 (*Continued*)

Group II. Addition of Whole Numbers with Proper Fractions and with Mixed Numbers

Type	Example	Description	Per Cent of Error	Rank of Difficulty
6	$5 + \frac{1}{4} = 5\frac{1}{4}$	Fraction added to a whole number.	11.0	9
7	$\frac{2}{3} + 7 = 7\frac{2}{3}$	Whole number added to a fraction.	5.5	2
8	$4 + 1\frac{2}{3} = 5\frac{2}{3}$	Mixed number added to a whole number.	12.0	10
9	$3\frac{1}{2} + 8 = 11\frac{1}{2}$	Whole number added to a mixed number. No reduction.	0.1	3
10	$2\frac{2}{4} + 4 = 6\frac{2}{4} = 6\frac{1}{2}$	Whole number added to a mixed number. Reduction.	15.9	17

Group III. Addition of Mixed Numbers with Fractions having Common Denominators

Type	Example	Description	Per Cent of Error	Rank of Difficulty
11	$3\frac{1}{5} + \frac{1}{5} = 3\frac{2}{5}$	Similar denominators. Sum of fractions is less than 1. No reduction.	10.3	7
12	$\frac{1}{4} + 2\frac{1}{4} = 2\frac{2}{4} = 2\frac{1}{2}$	Similar denominators. Sum of fractions is less than 1. Reduction.	15.7	16
13	$2\frac{1}{3} + 7\frac{2}{3} = 9\frac{3}{3} = 10$	Sum of fractions is 1. Carrying.	21.2	24.5
14	$3\frac{3}{5} + 2\frac{4}{5} = 5\frac{7}{5} = 6\frac{2}{5}$	Sum of fractions is more than 1. Carrying. No reduction.	26.5	28
15	$5\frac{5}{6} + 2\frac{5}{6} = 7\frac{10}{6} = 8\frac{2}{3}$	Sum of fractions is more than 1. Carrying. Reduction in fraction remaining.	28.8	33.5

Group IV. Addition of Proper Fractions with Related Denominators

Type	Example	Description	Per Cent of Error	Rank of Difficulty
16	$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$	One given denominator is common denominator. Sum is less than 1. No reduction.	10.8	8
17	$\frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$	One given denominator is common denominator. Sum is less than 1. Reduction.	17.8	20
18	$\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{4}{4} = 1$	One given denominator is common denominator. Reduction. Sum is 1.	14.4	15
19	$\frac{5}{6} + \frac{3}{4} + \frac{5}{12} = \frac{24}{12} = 2$	One given denominator is common denominator. Answer is whole number greater than 1.	18.2	21

TABLE 6, GROUP IV (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
20	$\frac{1}{3} + \frac{5}{6} = \frac{7}{6} = 1\frac{1}{6}$	One given denominator is common denominator. Sum is more than 1. No reduction in fraction remaining.	13.8	13
21	$\frac{7}{12} + \frac{3}{4} = \frac{16}{12} = 1\frac{4}{12} = 1\frac{1}{3}$	Sum is more than 1. Reduction in fraction remaining.	29.6	35
Group V. Addition of Mixed Numbers having Related Fractions and One Given Denominator as Common Denominator				
22	$3\frac{2}{3} + 2\frac{1}{6} = 5\frac{5}{6}$	Sum of fractions is less than 1. No reduction.	13.0	11
23	$1\frac{1}{4} + 3\frac{7}{12} = 4\frac{10}{12} = 4\frac{5}{6}$	Sum of fractions is less than 1. Reduction.	27.2	29
24	$1\frac{1}{4} + \frac{1}{2} + 2\frac{1}{4} = 3\frac{4}{4} = 4$	Sum of fractions is 1. Carrying.	21.2	24.5
25	$7\frac{1}{2} + 3\frac{3}{4} = 10\frac{5}{4} = 11\frac{1}{4}$	Sum of fractions is more than 1. Carrying. No reduction in fraction remaining.	21.0	23
26	$2\frac{2}{3} + 7\frac{5}{6} = 9\frac{9}{6} = 10\frac{3}{6} = 10\frac{1}{2}$	Sum of fractions is more than 1. Carrying. Reduction.	27.6	30.5
27	$3\frac{17}{24} + 4\frac{5}{6} + 3\frac{5}{8} = 10\frac{52}{24} = 12\frac{4}{24} = 12\frac{1}{6}$	Sum of fractions is more than 2. Carrying. Reduction in fraction remaining.	50.4	39
Group VI. Addition of Proper Fractions with Unrelated Denominators				
28	$\frac{1}{4} + \frac{1}{3} = \frac{7}{12}$	Common denominator is product of given denominators. Sum less than 1. No reduction.	9.1	5
29	$\frac{1}{3} + \frac{1}{5} + \frac{3}{10} = \frac{25}{30} = \frac{5}{6}$	Common denominator is product of two of the given denominators. Sum is less than 1. Reduction.	37.7	36
30	$\frac{3}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24} = 1\frac{1}{24}$	Common denominator is product of two of the given denominators. Sum of fractions more than 1. No reduction in fraction remaining.	27.0	30.5
31	$\frac{3}{5} + \frac{1}{2} + \frac{1}{4} = \frac{27}{20} = 1\frac{7}{20}$	Common denominator is product of two of the given denominators. Sum more than 1. No reduction in fraction remaining. Remaining fraction has numerator greater than 1.	24.3	27

TABLE 6, GROUP VI (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
32	$\frac{2}{3}+\frac{4}{5}+\frac{7}{10}=\frac{65}{30}=2\frac{5}{30}=2\frac{1}{6}$	Common denominator is product of two of the given denominators. Sum more than 2. Reduction in fraction remaining.	42.0	37

GROUP VII. ADDITION OF MIXED NUMBERS WITH UNRELATED FRACTIONS. COMMON DENOMINATOR IS PRODUCT OF UNLIKE DENOMINATORS

33	$4\frac{1}{5}+1\frac{1}{4}=5\frac{9}{20}$	Sum of fractions is less than 1. No reduction.	13.2	12
34	$1\frac{1}{6}+7\frac{3}{5}+3\frac{1}{6}=11\frac{28}{30}=11\frac{14}{15}$	Common denominator is product of two of given denominators. Sum of fractions is less than 1. Reduction.	55.0	40
35	$6\frac{2}{3}+4\frac{3}{4}=10\frac{17}{12}=11\frac{5}{12}$	Common denominator is product of given denominators. Sum of fractions is greater than 1. Carrying.	27.9	32
36	$5\frac{3}{5}+7\frac{1}{2}=12\frac{11}{10}=13\frac{1}{10}$	Common denominator is product of given denominators. Sum of fractions is greater than 1. Carrying. Reduction.	42.7	38

GROUP VIII.—ADDITION OF MIXED NUMBERS AND FRACTIONS

37	$5\frac{1}{12}+\frac{5}{6}=5\frac{11}{12}$	One given denominator is common denominator. Sum of fractions is less than 1. No reduction.	17.4	19
38	$\frac{5}{12}+1\frac{1}{3}=1\frac{9}{12}=1\frac{3}{4}$	One given denominator is common denominator. Sum of fractions is less than 1. Reduction.	28.8	33.5
39	$\frac{2}{3}+\frac{1}{4}+4\frac{1}{12}=5$	One given denominator is common denominator. Reduction. Sum of fractions is 1. Carrying.	19.1	22
40	$\frac{2}{5}+1\frac{2}{3}=1\frac{16}{15}=2\frac{1}{15}$	Common denominator is product of given denominators. Reduction. Sum of fractions is more than 1. Carrying.	24.2	26

TABLE 6 (*Continued*)

B. SUBTRACTIONS (53 Types)

GROUP I. PROPER FRACTIONS SUBTRACTED FROM PROPER AND IMPROPER FRACTIONS WITH SIMILAR DENOMINATORS

Type	Example	Description	Per Cent of Error	Rank of Difficulty
1	$\frac{5}{4} - \frac{3}{4} = \frac{2}{4} = \frac{1}{2}$	Similar denominators. Reduction.	5.0	3
2	$\frac{8}{9} - \frac{3}{9} = \frac{5}{9}$	Similar denominators. No reduction.	3.1	2
3	$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$	Similar denominators. No reduction.	2.8	1
4	$\frac{5}{8} - \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$	Similar denominators. Reduction.	6.9	5
5	$\frac{1}{2} - \frac{1}{2} = 0$	Fraction subtracted from fraction. Similar denominators. Remainder is zero.	29.0	33

GROUP II. PROPER FRACTIONS SUBTRACTED FROM MIXED NUMBERS. FRACTIONS WITH SIMILAR DENOMINATORS

Type	Example	Description	Per Cent of Error	Rank of Difficulty
6	$4\frac{2}{3} - \frac{1}{3} = 4\frac{1}{3}$	Similar denominators. Remainder is mixed number. No reduction.	21.5	17
7	$4\frac{2}{3} - \frac{2}{3} = 4$	Similar denominators. Fractions equal. Remainder is whole number. Zero difficulty.	40.2	53
8	$7\frac{5}{6} - \frac{1}{6} = 7\frac{4}{6} = 7\frac{2}{3}$	Similar denominators. Remainder is mixed number. Reduction.	29.2	35.5
9	$1\frac{1}{3} - \frac{2}{3} = \frac{2}{3}$	Fraction subtracted from mixed number. Similar denominators. Borrowing. Remainder is fraction. No reduction. Zero difficulty.	29.2	35.5

GROUP III. WHOLE NUMBERS SUBTRACTED FROM MIXED NUMBERS

Type	Example	Description	Per Cent of Error	Rank of Difficulty
10	$5\frac{1}{4} - 3 = 2\frac{1}{4}$	Remainder is mixed number.	8.2	7
11	$10\frac{1}{2} - 10 = \frac{1}{2}$	Whole numbers similar. Remainder is fraction. Zero difficulty.	15.4	18

GROUP IV. MIXED NUMBERS SUBTRACTED FROM MIXED NUMBERS WITH SIMILAR DENOMINATORS. NO BORROWING

Type	Example	Description	Per Cent of Error	Rank of Difficulty
12	$4\frac{3}{5} - 2\frac{1}{5} = 2\frac{2}{5}$	Mixed number subtracted from mixed number. Similar denominators. No reduction.	6.7	4

TABLE 6, GROUP IV (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
13	$3\frac{3}{4} - 2\frac{3}{4} = 1$	Similar denominators. Fractions equal. Remainder is whole number.	29.9	39
14	$7\frac{7}{8} - 2\frac{1}{8} = 5\frac{6}{8} = 5\frac{3}{4}$	Similar denominators. Remainder is mixed number. Reduction.	19.9	15
15	$1\frac{5}{8} - 1\frac{1}{8} = \frac{4}{8} = \frac{1}{2}$	Similar denominators. Remainder is fraction. Reduction. Zero difficulty.	14.8	11
16	$2\frac{1}{2} - 2\frac{1}{2} = 0$	Similar mixed numbers. Remainder is zero. Zero difficulty.	32.5	45
Group V. Mixed Numbers Subtracted from Whole Numbers, and Fractions Subtracted from Whole Numbers				
17	$1 - \frac{1}{2} = \frac{1}{2}$	Fraction subtracted from 1. Remainder is proper fraction. Borrowing. No reduction.	26.4	24
18	$9 - \frac{1}{2} = 8\frac{1}{2}$	Fraction subtracted from whole number. Borrowing. Remainder is mixed number. No reduction.	26.7	27
19	$5 - 1\frac{1}{3} = 3\frac{2}{3}$	Mixed number subtracted from whole number. Borrowing. Remainder is mixed number. No reduction.	28.6	32
20	$3 - 2\frac{5}{12} = \frac{7}{12}$	Mixed number subtracted from whole number. Borrowing. Remainder is fraction. No reduction. Zero difficulty.	38.0	52
Group VI. Proper Fractions and Mixed Numbers Subtracted from Mixed Numbers with Similar Denominators. Borrowing				
21	$6\frac{1}{3} - \frac{2}{3} = 5\frac{2}{3}$	Fraction subtracted from mixed number. Similar denominators. Borrowing. Remainder is mixed number. No reduction.	23.4	20
22	$1\frac{1}{8} - \frac{5}{8} = \frac{4}{8} = \frac{1}{2}$	Fraction subtracted from mixed number. Similar denominators. Borrowing. Remainder is proper fraction. Reduction. Zero difficulty.	31.8	44
23	$7\frac{1}{8} - \frac{5}{8} = 6\frac{4}{8} = 6\frac{1}{2}$	Fraction subtracted from mixed number. Similar denominators. Borrowing. Remainder is mixed number. Reduction.	31.8	44

TABLE 6, GROUP VI (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
24	$10\frac{1}{3} - 4\frac{2}{3} = 5\frac{2}{3}$	Mixed number subtracted from mixed number. Similar denominators. Borrowing. Remainder is mixed number. No reduction.	26.3	23
25	$9\frac{2}{5} - 8\frac{3}{5} = \frac{4}{5}$	Mixed number subtracted from mixed number. Similar denominators. Borrowing. Remainder is fraction. No reduction. Zero difficulty.	33.7	47
26	$4\frac{1}{8} - 1\frac{7}{8} = 2\frac{2}{8} = 2\frac{1}{4}$	Mixed number subtracted from mixed number. Similar denominators. Borrowing. Remainder is mixed number. Reduction.	33.9	48.5
27	$2\frac{1}{4} - 1\frac{3}{4} = \frac{2}{4} = \frac{1}{2}$	Mixed number subtracted from mixed number. Similar denominators. Borrowing. Remainder is fraction. Reduction. Zero difficulty.	27.3	28.5

GROUP VII. FRACTIONS SUBTRACTED FROM FRACTIONS; FRACTIONS FROM MIXED NUMBERS; MIXED NUMBERS FROM MIXED NUMBERS. RELATED FRACTIONS. NO BORROWING

Type	Example	Description	Per Cent of Error	Rank of Difficulty
28	$\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$	Related fractions. One given denominator is common denominator. No reduction.	7.8	6
29	$\frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$	Related fractions. One given denominator is common denominator. Remainder is fraction. Reduction.	13.2	10
30	$3\frac{3}{4} - \frac{1}{2} = 3\frac{1}{4}$	Fraction subtracted from mixed number. Related fractions. One given denominator is common denominator. Remainder is mixed number. No reduction.	14.9	12
31	$4\frac{5}{6} - \frac{1}{3} = 4\frac{3}{6} = 4\frac{1}{2}$	Fraction subtracted from mixed number. Related fractions. One given denominator is common denominator. Remainder is mixed number. Reduction.	26.5	25.5
32	$3\frac{3}{4} - 1\frac{1}{2} = 2\frac{1}{4}$	Mixed number subtracted from mixed number. Related fractions. Remainder is mixed number. No reduction. One given denominator is common denominator.	10.7	9

TABLE 6, GROUP VII (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
33	$1\frac{5}{6} - 1\frac{1}{2} = \frac{2}{6} = \frac{1}{3}$	Mixed number subtracted from mixed number. Related fraction. One given denominator is common denominator. Remainder is fraction. Reduction. Zero difficulty.	21.4	16
34	$7\frac{3}{5} - 1\frac{1}{10} = 6\frac{5}{10} = 6\frac{1}{2}$	Mixed number subtracted from mixed number. Related fractions. One given denominator is common denominator. Remainder is mixed number. Reduction.	25.4	22
35	$1\frac{3}{4} - 1\frac{1}{8} = \frac{5}{8}$	Mixed number subtracted from mixed number. Related fractions. One given denominator is common denominator. Remainder is proper fraction. No reduction. Zero difficulty.	21.8	18
36	$14\frac{3}{4} - 14\frac{1}{2} = \frac{1}{4}$	Mixed number subtracted from mixed number. Related fractions. Common denominator is denominator of fraction in minuend. Remainder is unit fraction. No reduction. Zero difficulty.	19.4	14

GROUP VIII. FRACTIONS AND MIXED NUMBERS SUBTRACTED FROM MIXED NUMBERS. BORROWING. RELATED FRACTIONS

Type	Example	Description	Per Cent of Error	Rank of Difficulty
37	$1\frac{1}{10} - \frac{4}{5} = \frac{3}{10}$	Fraction subtracted from mixed number. Related fractions. Borrowing. Remainder is fraction. No reduction.	28.6	32
38	$1\frac{1}{2} - \frac{5}{6} = \frac{4}{6} = \frac{2}{3}$	Fraction subtracted from mixed number. Borrowing. Remainder is fraction. Reduction.	30.1	40.5
39	$2\frac{1}{6} - \frac{2}{3} = 1\frac{3}{6} = 1\frac{1}{2}$	Fraction subtracted from mixed number. Related fractions. One given denominator is common denominator. Remainder is mixed number. Reduction.	37.2	51
40	$4\frac{1}{4} - \frac{5}{8} = 3\frac{5}{8}$	Fraction subtracted from mixed number. Related fractions. Common denominator is given denominator. Borrowing. Remainder is mixed number. No reduction.	27.8	30

TABLE 6, GROUP VIII (*Continued*)

Type	Example	Description	Per Cent of Cases	[illegible]
41	$7\frac{1}{2} - 2\frac{3}{4} = 4\frac{3}{4}$	Mixed number subtracted from mixed number. Related fractions. One given denominator is common denominator. Borrowing. Remainder is mixed number. No reduction.	[illegible]	[illegible]
42	$3\frac{1}{6} - 1\frac{2}{3} = 1\frac{3}{6} = 1\frac{1}{2}$	Mixed number subtracted from mixed number. Related fractions. One given denominator is common denominator. Borrowing. Remainder is mixed number. Reduction.	[illegible]	[illegible]
43	$3\frac{1}{2} - 2\frac{5}{8} = \frac{7}{8}$	Mixed number subtracted from mixed number. Related fractions. One given denominator is common denominator. Borrowing. Remainder is fraction. No reduction. Zero difficulty.	[illegible]	[illegible]
44	$9\frac{1}{10} - 8\frac{1}{2} = \frac{6}{10} = \frac{3}{5}$	Mixed number subtracted from mixed number. Related fractions. One given denominator is common denominator. Borrowing. Remainder is fraction. Reduction. Zero difficulty.	[illegible]	[illegible]

GROUP IX. FRACTIONS SUBTRACTED FROM FRACTIONS; FRACTIONS FROM MIXED NUMBERS. UNRELATED FRACTIONS

Type	Example	Description	Per Cent of Cases	[illegible]
45	$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$	Fraction subtracted from fraction. Unrelated. Common denominator is product of given denominators. No reduction.	[illegible]	[illegible]
46	$\frac{4}{15} - \frac{1}{6} = \frac{3}{30} = \frac{1}{10}$	Fraction subtracted from fraction. Unrelated. Common denominator is product of given denominators. Reduction.	[illegible]	[illegible]
47	$2\frac{3}{8} - \frac{1}{3} = 2\frac{1}{24}$	Fraction subtracted from mixed number. Unrelated. Common denominator is product of given denominators. Remainder is mixed number. No reduction. No borrowing.	[illegible]	[illegible]
48	$1\frac{1}{3} - \frac{1}{2} = \frac{5}{6}$	Fraction subtracted from mixed number. Unrelated. Common denominator is product of given denominators. Remainder is fraction. No reduction. Borrowing. Zero difficulty.	[illegible]	[illegible]

TABLE 6, GROUP IX (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
49	$8\frac{1}{2} - \frac{2}{3} = 7\frac{5}{6}$	Fraction subtracted from mixed number. Unrelated. Common denominator is product of given denominators. Borrowing. Remainder is mixed number. No reduction.	20.2	35.5
50	$4\frac{5}{6} - 1\frac{2}{5} = 3\frac{13}{30}$	Mixed number subtracted from mixed number. Unrelated. Common denominator is product of given denominators. No borrowing. Remainder is mixed number. No reduction.	23.7	21
51	$6\frac{1}{2} - 6\frac{2}{5} = \frac{1}{10}$	Mixed number subtracted from mixed number. Unrelated. Common denominator is product of given denominators. No borrowing. Remainder is fraction. No reduction.	22.5	19
52	$3\frac{1}{4} - 1\frac{2}{3} = 1\frac{7}{12}$	Mixed number subtracted from mixed number. Unrelated. Common denominator is product of given denominators. Remainder is mixed number. Borrowing. No reduction.	28.5	31
53	$6\frac{1}{4} - 5\frac{2}{3} = \frac{7}{12}$	Mixed number subtracted from mixed number. Unrelated. Common denominator is product of given denominators. Remainder is fraction. Borrowing. No reduction. Zero difficulty.	33.9	48.5

C. MULTIPLICATION (45 Types)

Group I. Proper Fractions Multiplied by Whole Numbers

1	$\frac{1}{4} \times 8 = 2$	Whole number multiplied by fraction. Cancellation. Product is whole number.	7.3	2
2	$\frac{1}{6} \times 3 = \frac{1}{2}$	Whole number multiplied by fraction. Cancellation. Product is unit fraction.	12.8	5
3	$\frac{2}{5} \times 2 = \frac{4}{5}$	Whole number multiplied by fraction. No cancellation. Product is proper fraction.	13.5	6

TABLE 6, GROUP I (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
4	$\frac{2}{3} \times 16 = 10\frac{2}{3}$	Whole number multiplied by fraction. No cancellation. Product is mixed number.	24.8	21
5	$\frac{5}{6} \times 14 = 11\frac{2}{3}$	Whole number multiplied by fraction. Cancellation. Product is mixed number.	26.4	26

GROUP II. WHOLE NUMBERS MULTIPLIED BY PROPER FRACTIONS

6	$2 \times \frac{1}{3} = \frac{2}{3}$	Whole number multiplied by proper fraction.	17.9	14
7	$3 \times \frac{1}{3} = 1$	Whole number multiplied by fraction. Cancellation. Product is 1.	9.8	3
8	$4 \times \frac{1}{2} = 2$	Whole number multiplied by fraction. Cancellation. Product is whole number.	10.9	4
9	$12 \times \frac{1}{5} = 2\frac{2}{5}$	Whole number multiplied by fraction. No cancellation. Product is mixed number. No reduction.	16.8	9
10	$8 \times \frac{1}{6} = 1\frac{1}{3}$	Whole number multiplied by fraction. Cancellation. Product is mixed number.	18.2	15

GROUP III. PROPER FRACTIONS MULTIPLIED BY PROPER FRACTIONS INVOLVING ALSO UNIT FRACTIONS

11	$\frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$	Unit fraction multiplied by unit fraction. No cancellation. Product is unit fraction. No reduction.	14.2	7
12	$\frac{1}{2} \times \frac{4}{5} = \frac{2}{5}$	Proper fraction multiplied by unit fraction. Cancellation. Product is proper fraction.	15.5	8
13	$\frac{1}{2} \times \frac{7}{10} = \frac{7}{20}$	Proper fraction multiplied by unit fraction. No cancellation. Product is proper fraction. No reduction.	6.7	1
14	$\frac{5}{9} \times \frac{3}{7} = \frac{5}{21}$	Proper fraction multiplied by proper fraction. Cancellation. Product is proper fraction.	19.7	17.5

TABLE 6 (*Continued*)

GROUP IV. WHOLE NUMBERS MULTIPLIED BY MIXED NUMBERS

Type	Example	Description	Per Cent of Error	Rank of Difficulty
15	$6 \times 2\frac{1}{3} = 14$	Whole number multiplied by mixed number. Cancellation. Product is whole number.	19.7	17.5
16	$7 \times 4\frac{1}{2} = 31\frac{1}{2}$	Whole number multiplied by mixed number. No cancellation. Carrying. Product is mixed number. No reduction.	28.1	27
17	$6 \times 3\frac{3}{4} = 22\frac{1}{2}$	Whole number multiplied by mixed number. Cancellation. Product is mixed number. No reduction.	29.3	31
18	$2 \times 4\frac{1}{8} = 8\frac{2}{8}$	Whole number multiplied by mixed number. No cancellation. No reduction. Product is mixed number.	26.0	24

GROUP V. MIXED NUMBERS MULTIPLIED BY WHOLE NUMBERS

Type	Example	Description	Per Cent of Error	Rank of Difficulty
19	$3\frac{1}{5} \times 2 = 6\frac{2}{5}$	Mixed number multiplied by whole number. No cancellation. Product is mixed number. No reduction.	38.3	28.5
20	$3\frac{1}{6} \times 4 = 12\frac{2}{3}$	Mixed number multiplied by whole number. Cancellation. Product is mixed number. No reduction.	33.0	33
21	$3\frac{2}{3} \times 9 = 33$	Mixed number multiplied by whole number. Cancellation. Product is whole number.	19.4	16

GROUP VI. PROPER FRACTIONS MULTIPLIED BY MIXED NUMBERS

Type	Example	Description	Per Cent of Error	Rank of Difficulty
22	$\frac{1}{4} \times 2\frac{1}{2} = \frac{5}{8}$	Unit fraction multiplied by mixed number. No cancellation. Product is proper fraction. No reduction.	17.1	10
23	$\frac{1}{2} \times 1\frac{1}{5} = \frac{3}{5}$	Unit fraction multiplied by mixed number. Cancellation. Product is proper fraction. No reduction.	17.7	12.5
24	$\frac{5}{8} \times 6\frac{2}{5} = 4$	Proper fraction multiplied by mixed number. Cancellation. Product is whole number.	17.3	11
25	$\frac{2}{5} \times 3\frac{2}{3} = 1\frac{7}{15}$	Proper fraction multiplied by mixed number. No cancellation. Product is mixed number. No reduction.	26.1	25

TABLE 6 (*Continued*)

GROUP VII. MIXED NUMBERS MULTIPLIED BY PROPER FRACTIONS

TYPE	EXAMPLE	DESCRIPTION	PER CENT OF ERROR	RANK OF DIFFICULTY
26	$3\frac{1}{3} \times \frac{2}{9} = \frac{20}{27}$	Mixed number multiplied by proper fraction. No cancellation. Product is proper fraction. No reduction.	21.3	19
27	$3\frac{1}{3} \times \frac{1}{8} = \frac{5}{12}$	Mixed number multiplied by unit fraction. Cancellation. Product is proper fraction. No reduction.	28.9	30
28	$7\frac{1}{2} \times \frac{2}{5} = 3$	Mixed number multiplied by proper fraction. Cancellation. Product is whole number.	17.7	12.5
29	$6\frac{1}{5} \times \frac{3}{7} = 2\frac{23}{35}$	Mixed number multiplied by proper fraction. No cancellation. Product is mixed number. No reduction.	42.0	37
30	$2\frac{1}{4} \times \frac{1}{4} = \frac{9}{16}$	Mixed number multiplied by unit fraction. No cancellation. Product is proper fraction. No reduction.	31.5	32

GROUP VIII. MIXED NUMBERS MULTIPLIED BY MIXED NUMBERS

TYPE	EXAMPLE	DESCRIPTION	PER CENT OF ERROR	RANK OF DIFFICULTY
31	$2\frac{1}{3} \times 2\frac{1}{2} = 5\frac{5}{6}$	Mixed number multiplied by mixed number. No cancellation. Product is mixed number.	23.3	28.5
32	$6\frac{1}{4} \times 6\frac{2}{5} = 40$	Mixed number multiplied by mixed number. Cancellation. Product is whole number.	22.3	20
33	$3\frac{3}{4} \times 3\frac{1}{3} = 12\frac{1}{2}$	Mixed number multiplied by mixed number. Cancellation. Product is mixed number. No reduction.	33.7	34
34	$3\frac{2}{7} \times 3\frac{2}{3} = 12\frac{1}{21}$	Mixed number multiplied by mixed number. No cancellation. Product is mixed number. No reduction.	45.7	38

GROUP IX. SELECTED COMBINATION TYPES INVOLVING MULTIPLICATION OF WHOLE NUMBERS, MIXED NUMBERS, AND FRACTIONS

TYPE	EXAMPLE	DESCRIPTION	PER CENT OF ERROR	RANK OF DIFFICULTY
35	$\frac{5}{8} \times \frac{16}{25} \times \frac{1}{2} = \frac{1}{5}$	Multiplication of three proper fractions. Cancellation.	24.5	22.5
36	$\frac{2}{3} \times \frac{3}{4} \times 4\frac{1}{2} = 2\frac{1}{4}$	Multiplication of two proper fractions and a mixed number. Cancellation.	53.4	39

TABLE 6, GROUP IX (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
37	½ × 10 × ⅗ = 3	Multiplication of two proper fractions and a whole number. Complete cancellation.	24.5	22.5
38	⅜ × 6 × 2⅓ = 5¼	Multiplication of a proper fraction, a whole number, and a mixed number. Cancellation.	38.1	36
39	4 × 3⅗ × 7 = 100⅘	Multiplication of two whole numbers and a mixed number. No cancellation.	56.5	43
40	4½ × 2⅔ × 1⅞ = 22½	Multiplication of three mixed numbers. Cancellation.	55.2	41

GROUP X. COMBINATION OF DIFFICULT TYPES OF MULTIPLICATION INVOLVING LARGE NUMBERS

41	24 3⅛ —— 3 72 —— 75	Whole number multiplied by mixed number. Whole number multiplied by unit fraction. Product is whole number. Carrying whole number. Final product is whole number.	36.5	35
42	36⅚ 12 —— 10 72 36 —— 442	Mixed number multiplied by whole number. Proper fraction multiplied by whole number. Product is whole number. Carrying whole number. Final product is whole number.	54.0	40
43	18 15¾ —— 13½ 90 18 —— 283½	Whole number multiplied by mixed number. Whole number multiplied by proper fraction. Product is mixed number. Carrying mixed number. Final product is mixed number.	60.0	45
44	25½ 27 —— 13½ 175 50 —— 688½	Mixed number multiplied by whole number. Unit fraction multiplied by whole number. Product is mixed number. Carrying mixed number. Final product is mixed number.	56.1	42

TABLE 6, GROUP X (*Continued*)

Example	Description	Per Cent of Error	Rank of Difficulty
23 $18\frac{4}{7}$ $13\frac{1}{7}$ 184 23 $427\frac{1}{7}$	Whole number multiplied by mixed number. Whole number multiplied by proper fraction. Product is mixed number. Carrying mixed number. Final product is mixed number.	59.7	44

D. DIVISION (40 Types)

Group I. Whole Numbers Divided by Proper Fractions

1	$5 \div \frac{1}{2} = 10$	Whole number divided by unit fraction. Quotient is whole number.	16.1	7
2	$5 \div \frac{2}{3} = 7\frac{1}{2}$	Whole number divided by proper fraction. No cancellation.	14.6	4
3	$12 \div \frac{3}{4} = 16$	Whole number divided by proper fraction. Cancellation. Quotient is whole number.	21.3	14
4	$14 \div \frac{4}{5} = 17\frac{1}{2}$	Whole number divided by proper fraction. Cancellation. Quotient is mixed number.	25.9	22
5	$2 \div \frac{4}{5} = 2\frac{1}{2}$	Whole number divided by proper fraction. Cancellation. Quotient is mixed number.	19.5	9

Group II. Fraction Divided by Fraction

6	$\frac{1}{7} \div \frac{1}{6} = \frac{6}{7}$	Unit fraction divided by unit fraction. No cancellation. Quotient is proper fraction.	9.4	1
7	$\frac{1}{6} \div \frac{1}{3} = \frac{1}{2}$	Unit fraction divided by unit fraction. Cancellation. Quotient is unit fraction.	14.3	3
8	$\frac{1}{3} \div \frac{1}{6} = 2$	Unit fraction divided by unit fraction. Cancellation. Quotient is whole number more than 1.	15.1	5
9	$\frac{3}{8} \div \frac{3}{8} = 1$	Proper fraction divided by proper fraction. Cancellation. Quotient is 1.	13.4	2

TABLE 6, GROUP II (*Continued*)

Type	Examples	Description	Per Cent of Error	Rank of Difficulty
10	$\frac{4}{5} \div \frac{1}{2} = 1\frac{3}{5}$	Proper fraction divided by unit fraction. No cancellation. Quotient is mixed number. No reduction.	23.9	16
11	$\frac{4}{5} \div \frac{3}{10} = 2\frac{2}{3}$	Proper fraction divided by proper fraction. Cancellation. Quotient is mixed number. No reduction.	41.8	38.5
Group III. Mixed Number Divided by a Proper Fraction				
12	$1\frac{1}{4} \div \frac{1}{3} = 3\frac{3}{4}$	Mixed number divided by unit fraction. No cancellation. Quotient is mixed number. No reduction.	19.4	8
13	$1\frac{1}{2} \div \frac{1}{2} = 3$	Mixed number divided by unit fraction. Cancellation. Quotient is whole number.	15.8	6
14	$1\frac{3}{8} \div \frac{3}{10} = 4\frac{7}{12}$	Mixed number divided by proper fraction. Cancellation. Quotient is mixed number.	36.8	37
Group IV. Proper Fraction Divided by Whole Number				
15	$\frac{1}{5} \div 5 = \frac{1}{25}$	Unit fraction divided by whole number. Quotient is unit fraction.	42.2	40
16	$\frac{5}{6} \div 4 = \frac{5}{24}$	Proper fraction divided by whole number. No cancellation. Quotient is proper fraction.	20.5	11.5
17	$\frac{5}{6} \div 5 = \frac{1}{6}$	Proper fraction divided by whole number. Cancellation. Product is unit fraction.	21.2	13
18	$\frac{4}{5} \div 6 = \frac{2}{15}$	Proper fraction divided by whole number. Cancellation. Product is proper fraction.	29.2	28
Group V. Mixed Number Divided by Whole Number				
19	$1\frac{1}{3} \div 5 = \frac{4}{15}$	Mixed number divided by whole number. No cancellation. Quotient is proper fraction.	21.4	15
20	$1\frac{1}{5} \div 4 = \frac{3}{10}$	Mixed number divided by whole number. Cancellation. Quotient is proper fraction.	29.3	29

TABLE 6, GROUP V (*Continued*)

Type	Example	Description	Per Cent of Error	Rank of Difficulty
21	$5\frac{2}{3} \div 5 = 1\frac{2}{15}$	Mixed number divided by whole number. No cancellation. Quotient is mixed number.	26.7	25.5
22	$5\frac{3}{5} \div 2 = 2\frac{4}{5}$	Mixed number divided by whole number. Cancellation. Quotient is mixed number.	31.4	31

GROUP VI. PROPER FRACTION DIVIDED BY MIXED NUMBER

Type	Example	Description	Per Cent of Error	Rank of Difficulty
23	$\frac{1}{3} \div 1\frac{1}{2} = \frac{2}{9}$	Unit fraction divided by mixed number. No cancellation. Quotient is proper fraction.	32.7	35.5
24	$\frac{3}{4} \div 1\frac{2}{3} = \frac{9}{20}$	Proper fraction divided by mixed number. No cancellation. Quotient is proper fraction.	25.8	20
25	$\frac{5}{12} \div 1\frac{5}{6} = \frac{5}{22}$	Proper fraction divided by mixed number. Cancellation. Quotient is proper fraction.	32.3	34
26	$\frac{1}{6} \div 1\frac{1}{2} = \frac{1}{9}$	Unit fraction divided by mixed number. Cancellation. Quotient is unit fraction.	27.0	27

GROUP VII. WHOLE NUMBER DIVIDED BY MIXED NUMBER

Type	Example	Description	Per Cent of Error	Rank of Difficulty
27	$2 \div 2\frac{1}{2} = \frac{4}{5}$	Whole number divided by mixed number. No cancellation. Quotient is proper fraction.	24.7	17
28	$3 \div 4\frac{1}{2} = \frac{2}{3}$	Whole number divided by mixed number. Cancellation. Quotient is proper fraction.	24.8	18
29	$5 \div 2\frac{3}{4} = 1\frac{9}{11}$	Whole number divided by mixed number. No cancellation. Quotient is mixed number.	32.7	35.5
30	$5 \div 2\frac{1}{2} = 2$	Whole number divided by mixed number. Cancellation. Quotient is whole number.	26.6	24
31	$12 \div 6\frac{3}{5} = 1\frac{9}{11}$	Whole number divided by mixed number. Cancellation. Quotient is mixed number.	25.8	20

Group VIII. Mixed Number Divided by Mixed Number

Type	Example	Description	Per Cent of Error	Rank of Difficulty
32	$1\frac{[illegible]}{} \div 3\frac{[illegible]}{} =$ [illegible]	Mixed number divided by mixed number. No cancellation. Quotient is proper fraction.	26.7	25.5
33	$1\frac{2}{3} \div 3\frac{1}{3} = \frac{1}{2}$	Mixed number divided by mixed number. Cancellation. Quotient is proper fraction.	31.7	32
34	$2\frac{3}{8} \div 2\frac{3}{8} = 1$	Mixed number divided by mixed number. Cancellation. Quotient is 1.	20.5	11.5
35	$3\frac{1}{3} \div 1\frac{3}{4} = 1\frac{19}{21}$	Mixed number divided by mixed number. No cancellation. Quotient is mixed number.	41.8	38.5
36	$3\frac{3}{8} \div 1\frac{1}{4} = 2\frac{7}{10}$	Mixed number divided by mixed number. Cancellation. Quotient is mixed number.	30.7	30
37	$4\frac{1}{2} \div 2\frac{1}{4} = 2$	Mixed number divided by mixed number. Cancellation. Quotient is whole number more than 1.	32.1	33

Group IX. Whole Number Divided by Whole Number

Type	Example	Description	Per Cent of Error	Rank of Difficulty
38	$9 \div 8 = 1\frac{1}{8}$	Whole number divided by whole number. No cancellation. Quotient is mixed number.	19.9	10
39	$10 \div 6 = 1\frac{2}{3}$	Whole number divided by whole number. Cancellation. Quotient is mixed number.	25.8	20
40	$2 \div 14 = \frac{1}{7}$	Whole number divided by whole number. Cancellation. Quotient is proper fraction.	26.3	23

E. SUMMARY

Process	Range	Median	Mean
Addition..............	5.0 – 55.0	18.0	20.2
Subtraction...........	2.8 – 40.2	26.8	25.6
Multiplication.........	6.7 – 60.0	25.3	28.2
Division..............	9.4 – 42.2	25.3	25.8

There was a large variation in the per cents of error on the examples in each of the four processes in fractions. This shows that there is a wide range in the difficulty of the various types of examples in each process. According to Part E of Table 6 the range of difficulty is greatest in multiplication and least in division. There is not much difference in the median and mean per cents of error.

The means vary from 20.2 per cent for addition to 28.2 per cent for multiplication.

The medians vary from 18.0 per cent for addition to 26.8 per cent for subtraction.

There is every reason to believe that efficient practice in fractions must be characterized by a careful distribution of practice and instruction on the various types of examples in each process. The above lists of types should not be considered as final or complete. The data do not show what the per cents of error might be on other examples similar in type to those given in the table. No data are available to show what the difficulty of these types would be if the instructional material had been efficiently organized. It seems reasonable to assume that the high per cents of error on certain apparently simple types involving special difficulties, such as zeros in subtraction, may be due to the fact that such examples have not occurred in the practice provided by current instructional materials. Careful consideration must be given by the teacher to the development of the skills and abilities for each process listed on the preceding pages.

No data are available for measuring the amount of transfer from simple types to more complex types. Undoubtedly there is considerable. However, until such data are available, teachers must make certain that

pupils receive well distributed practice on all important types of examples, rather than on only a few selected at random. The assumption that the transfer from type to type is complete is unsafe.

3. A Special Study of Faults in Work in Fractions. The diagnostic tests in fractions which have just been described were given to two hundred pupils in each of Grades 5A, 6B, and 6A in six elementary schools in Minneapolis. A careful study was next made of the examples they worked incorrectly to discover, if possible, the causes of errors. Table 7 shows the total number of errors that were analyzed.[1]

TABLE 7

Number of Errors Analyzed in Examples in Addition, Subtraction, Multiplication, and Division of Fractions

Grade	Addition	Subtraction	Multiplication	Division	Total
5A	2,695	3,613	....	....	6,308
6B	1,477	1,978	1,237	2,348	7,040
6A	2,030	1,920	1,240	2,527	7,717
Total	6,202	7,511	2,477	4,875	21,065

The total number of errors analyzed was 21,065.[2] The largest number of errors in a single process was in subtraction; then follow in order addition, division, and multiplication. The pupils in grade 5A were not tested in multiplication and division because they had not had enough work in these processes. It is interesting to note

[1] The tables that follow have previously appeared in an article by the author, "An Analysis of Errors in Fractions," *Elementary School Journal*, Vol. 28, pp. 760–70.

[2] Abbie Chestek, graduate student at the University of Minnesota, assisted in this investigation.

that the errors in subtraction decrease grade by grade, while the total for addition is much larger in grade 6A than in grade 6B, although somewhat smaller in grade 6A than in grade 5A. The results of tests given in grades 7 and 8 show that the pupils in these grades make many errors and clearly need additional practice in solving fractions.

The detailed analysis of the errors made in working 24,000 examples in the addition of fractions, 27,000 examples in subtraction, 18,000 examples in multiplication, and 14,800 examples in division is the basis of the data in the tables on errors in fractions. The total number of trials of each type of example in addition was 600; in subtraction, 600; in multiplication, 400; and in division, 400. The total number of examples analyzed was 83,800.

Morton[1] reports a similar analysis of 1029 errors in the four processes in proper fractions, not including mixed numbers, by only 36 eighth-grade pupils. While his study supplies valuable data, it gives no information on many of the important difficulties in working with mixed numbers and it does not reveal the major causes or types of difficulty in grades where these processes are first being taught. His classification of errors is also quite general.

4. How to Determine the Types of Errors in Fractions.

The technique that was used to determine the nature of the difficulty which caused the pupil to work a given example incorrectly was to examine his written work carefully and by analysis to locate the cause of error. This method can be readily explained by means of the

[1] R. L. Morton, "An Analysis of Pupils' Errors in Fractions," *Journal of Educational Research*, Vol. 9 pp. 117–25.

following examples, which illustrate some of the most common types of errors in each of the processes:

Types of Addition Errors

(*a*) $\frac{2}{3} + \frac{3}{6} = \frac{5}{9}$

Here the pupil added the numerators (2 and 3) and added the denominators (3 and 6) to find the terms 5 and 9, respectively, in the answer.

(*b*) $\frac{2}{3} + \frac{3}{6} = \frac{5}{6}$

Here the pupil added the numerators without changing the form of the first fraction and used the common denominator (6) as the denominator of the answer.

(*c*) $5\frac{3}{5} + 7\frac{9}{8} = 13\frac{14}{16}$

All of the work was correct except for the failure to reduce the fraction in the sum.

(*d*) $7\frac{3}{4} + 3\frac{1}{2} = 10\frac{5}{4}$

Here the pupil failed to reduce the mixed number in the sum to the simplest form.

Typical Subtraction Errors

(*a*) $\frac{5}{8} - \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$

The pupil used the wrong process.

(*b*) $10\frac{1}{4} - 3\frac{2}{4} = 7\frac{3}{4}$

The pupil disregarded having borrowed from 10.

(*c*) $3\frac{1}{8} - \frac{6}{8} = 2\frac{11}{8} - \frac{6}{8} = 2\frac{5}{8}$

The pupil prefixed the number borrowed to the 1 in the fraction, changing $3\frac{1}{8}$ to $2\frac{11}{8}$.

(*d*) $3 - 1\frac{2}{3} = 2\frac{2}{3}$

The pupil subtracted the two integers, and placed the same fraction in the answer.

Typical Multiplication Errors

(*a*) $\frac{\cancel{5}}{\underset{3}{\cancel{6}}} \times \frac{\cancel{3}}{\underset{2}{\cancel{10}}} = \frac{1}{6}$

Here there is an error in division of 6 by 3.

(*b*) $\frac{5}{6} \times 14 = \frac{58}{6} = 9\frac{2}{3}$

Here there is an error in multiplication.

(*c*) $6 \times 2\frac{1}{3} = 6 \times \frac{3}{7} = \frac{18}{7} = 2\frac{4}{7}$

Here the pupil inverted the multiplicand, evidently confusing multiplication with division.

(*d*) $3 \times \frac{4}{15} = \frac{12}{15}$

Failure to cancel or reduce to lowest terms.

Typical Division Errors

(*a*) $1\frac{3}{8} \div 1\frac{2}{3} = \frac{11}{8} \times \frac{5}{3} = \frac{55}{24} = 2\frac{7}{24}$

Used the wrong process.

(*b*) $1\frac{1}{5} \div 3\frac{1}{2} = \frac{5}{6} \times \frac{7}{2} = \frac{35}{12} = 2\frac{7}{12}$

Inverted the dividend instead of the divisor.

(*c*) $1\frac{3}{8} \div 1\frac{2}{3} = \frac{11}{8} \times \frac{3}{5} = \frac{33}{40} = 1\frac{7}{33}$

Divided denominator of fraction by the numerator.

$$(d)\ 3\frac{1}{3} \div 1\frac{3}{4} = \frac{\overset{5}{\cancel{10}}}{3} \times \frac{4}{\underset{6}{\cancel{12}}} = \frac{20}{18} = 1\frac{1}{9}$$

Difficulty in reducing mixed number to improper fraction.

The tables in this chapter contain many other typical errors in fractions, which can be discovered by an analysis of the work on the pupil's paper. In many cases there are peculiar errors, the reasons for which cannot be determined by this method. For instance, the cause of the error in the example, which was worked $2\frac{2}{3} + 2\frac{1}{6} = 6\frac{1}{6}$, is not apparent. It can be determined, however, by asking the pupil to give the steps in the solution orally, the same technique that has been described for whole numbers. As can be seen from the tables that follow, the proportion of errors for which the causes cannot be located by an examination of the work on the paper is relatively small for subtraction, multiplication, and divi-

sion. The proportion for addition is quite substantial, namely, 20.4 per cent of all errors. Clearly a diagnosis of the causes of difficulty in addition of fractions must embody the results of an analysis of the work on the paper and also the checking up by the oral method on many of the less obvious difficulties.

5. THE MOST COMMON FAULTS IN FRACTIONS.

A knowledge of the most common faults in fractions serves a number of purposes. The teacher can consciously try to prevent the occurrence of these typical errors by a more careful teaching of the particular elements which appear to cause the most difficulty in each process. Drill materials can be prepared which provide special types of drill and practice in the difficult steps. Sometimes, in spite of careful teaching and the use of well organized practice exercises, there are pupils who have difficulty. A knowledge of the known chief sources of error in each process will facilitate diagnosis of the cause of the difficulty and aid in the assignment of the proper remedial exercises.

(a) **Addition of fractions.** The most common faults or sources of error in addition of fractions in grades 5 and 6 are given in Table 8 together with illustrations of each kind of error.

TABLE 8

ANALYSIS OF ERRORS IN THE ADDITION OF FRACTIONS

	GRADE V A	GRADE VI B	GRADE VI A	TOTAL	PER CENT
1. Lack of comprehension of process involved	298	375	581	1,254	20.2
(a) Added denominators and numerators: $\frac{2}{3}+\frac{3}{6}=\frac{5}{9}$	64	131	281	476	....
(b) Added numerators and multiplied denominators: $\frac{4}{5}+\frac{2}{5}=\frac{6}{25}$	26	69	53	148	....

TABLE 8 (*Continued*)

	Grade V A	Grade VI B	Grade VI A	Total	Per Cent
(*c*) Added numerators without changing fractions to common denominator; used one of the two denominators for denominator in sum: $\frac{2}{3}+\frac{3}{6}=\frac{5}{6}$	192	167	221	580	....
(*d*) Added numerators for denominator and added denominators for numerator: $\frac{2}{8}+\frac{4}{8}=\frac{16}{6}=2\frac{2}{3}$	0	0	15	15	....
(*e*) Multiplied numerator and denominator for numerator: $\frac{3}{4}+\frac{3}{4}=\frac{12}{4}+\frac{12}{4}=\frac{24}{4}=6$	3	2	6	11	....
(*f*) Multiplied numerators and added denominators: $\frac{1}{6}+\frac{2}{3}=\frac{2}{9}$	13	6	5	24	....
2. Difficulty in reducing fractions to lowest terms	658	200	230	1,088	17.5
(*a*) Did not reduce fraction: $5\frac{3}{8}+7\frac{7}{8}=13\frac{14}{16}$	593	150	169	912	....
(*b*) Divided denominator by numerator: $\frac{1}{2}+\frac{1}{6}=\frac{4}{6}=1\frac{1}{3}$	42	14	39	95	....
(*c*) Divided denominator and numerator by different numbers: $\frac{18}{30}=\frac{3}{2}$	23	36	22	81	....
3. Difficulty with improper fractions	489	250	322	1,061	17.1
(*a*) Did not change improper fraction to mixed number: $7\frac{3}{4}+3\frac{2}{4}=10\frac{5}{4}$	313	152	243	708	....
(*b*) Changed improper fraction but did not add to whole number: $2\frac{1}{3}+7\frac{2}{3}=9\frac{3}{3}=1$	176	98	79	353	....
4. Computation errors	302	265	288	855	13.8
(*a*) Addition: $3\frac{1}{2}+8=13\frac{1}{2}$	68	81	131	280	....
(*b*) Subtraction: $\frac{5}{6}+\frac{5}{8}=\frac{20}{24}+\frac{15}{24}=\frac{35}{24}=1\frac{12}{24}$	38	22	9	69	....
(*c*) Division: $\frac{2}{3}+\frac{5}{6}=\frac{9}{6}=1\frac{1}{3}$	33	4	13	50	....
(*d*) Unknown: $\frac{3}{8}+1\frac{2}{3}=1\frac{11}{18}$	163	158	135	456	....
5. Omitted example (no attempt)	83	26	58	167	2.7
6. Used wrong process	82	54	20	156	2.5
(*a*) Subtraction: $\frac{5}{8}+\frac{1}{4}=\frac{4}{8}=\frac{1}{2}$	52	19	12	83	....
(*b*) Multiplication: $\frac{1}{3}+\frac{2}{5}=\frac{2}{15}=\frac{1}{8}$	12	1	2	15	....
(*c*) Subtracted fractions and added whole numbers: $3\frac{2}{3}+2\frac{1}{6}=5\frac{3}{6}=5\frac{1}{2}$	3	31	1	35	....
(*d*) Added fractions and multiplied whole numbers: $3\frac{1}{2}+2\frac{1}{2}=6\frac{2}{4}=7$	4	3	3	10	....
(*e*) Added fractions and subtracted whole numbers: $6\frac{2}{3}+4\frac{3}{4}=2\frac{17}{12}=3\frac{5}{12}$	11	0	2	13	....

TABLE 8 (*Continued*)

	Grade V A	Grade VI B	Grade VI A	Total	Per Cent
7. Partial operation	54	35	47	136	2.2
(*a*) Added fractions but disregarded whole numbers: $1\frac{1}{4} + 3\frac{3}{4} = \frac{4}{4} = 1$	40	35	44	119	....
(*b*) Added whole numbers but disregarded fractions: $3\frac{2}{3}+2\frac{1}{6}=5$	14	0	3	17	....
8. Difficulty in changing fractions to common denominator	28	19	47	94	1.5
(*a*) Changed to wrong denominator: $\frac{5}{8} + \frac{1}{4} = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$	0	11	10	21	....
(*b*) Did not multiply numerator in reduction: $\frac{2}{3} + \frac{3}{6} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$	2	2	6	10	....
(*c*) Did not express denominator: $\frac{1}{2} + \frac{1}{5} = 7$	5	0	1	6	....
(*d*) Added common denominator to numerator of fraction changed: $\frac{1}{3} + \frac{2}{6} = \frac{12}{6} + \frac{2}{6} = 1\frac{2}{6}$	21	6	30	57	....
9. Difficulty in borrowing	15	23	33	71	1.1
(*a*) In adding whole number and mixed number, borrowed from whole number, added fractions, and left improper fraction in sum: $2\frac{3}{4} + 4 = 2\frac{3}{4} + 3\frac{4}{4} = 5\frac{7}{4}$	15	18	33	66	....
(*b*) In adding whole number and mixed number, borrowed from whole number, subtracted fractions, and added whole numbers: $4 + 1\frac{2}{3} = 3\frac{3}{3} + 1\frac{2}{3} = 4\frac{1}{3}$	0	5	0	5	....
10. Difficulty with proper fractions: $\frac{1}{4} + \frac{3}{4} = \frac{5}{4} = 1\frac{1}{4}$	11	6	21	38	0.6
11. Errors in copying: $7 + \frac{3}{4} = 7\frac{3}{8}$	4	6	4	14	0.2
12. Difficulty unknown: $2\frac{2}{3} + 2\frac{1}{6} = 2\frac{4}{6} + 2\frac{1}{6} = 6\frac{1}{6}$	671	218	379	1,268	20.4
Total	2,695	1,477	2,030	6,202	99.8

The faults in Table 8 are grouped under twelve headings. The largest group of errors, 20.2 per cent of the total, is due to lack of comprehension of the process involved. The most common fault in this group, 580 errors in all, is found in such solutions as, $\frac{2}{3} + \frac{3}{6} = \frac{5}{6}$, in which the pupil neglected to change the form of the fraction, $\frac{2}{3}$, added the numerators, 2 and 3, and wrote the common denominator as the denominator of the sum.

There were almost as many errors, 476 in all, of the type, $\frac{2}{3} + \frac{3}{6} = \frac{5}{9}$, in which the pupil added both numerators and denominators to get the answer. Both of these errors were common to all three of the grades that were studied.

Difficulty in reduction of fractions constituted the next largest source of error in addition. Failure to reduce proper fractions in mixed numbers, such as $13\frac{14}{16}$, or improper fractions in answers, such as $10\frac{5}{4}$, were the chief faults. There is also considerable difficulty in changing such numbers as $9\frac{3}{3}$ to 10, the most common fault being that $\frac{3}{3}$ is changed to 1, and the answer is written $9\frac{3}{3} = 1$. While many of these errors may be due to carelessness, nevertheless, the results suggest the advisability of considerable practice in reduction.

Many errors in addition—13.8 per cent of the total—are directly due to faulty computations. Others are due to incomplete work, difficulty in changing fractions to a common denominator, use of the wrong process in whole or in part, and peculiar errors found less frequently. It is probable that reducing fractions to a common denominator may be a greater source of difficulty than is apparent from the results of this study, since the denominators in the examples in the diagnostic test are simple, and in almost all of the examples only two fractions were involved.

(b) **Subtraction of fractions.** The most common faults in subtraction of fractions are given in Table 9.

TABLE 9

ANALYSIS OF ERRORS IN THE SUBTRACTION OF FRACTIONS

	Grade V A	Grade VI B	Grade VI A	Total	Per Cent
1. Difficulty in borrowing	769	414	645	1,828	24.3
(a) Disregarded having borrowed from whole number: $10\frac{1}{4} - 3\frac{3}{4} = 7\frac{2}{4}$	333	195	239	767	....

TABLE 9 (*Continued*)

	Grade V A	Grade VI B	Grade VI A	Total	Per Cent
1. Difficulty in borrowing (*Continued*)					
(*b*) Prefixed number borrowed to numerator: $3\frac{1}{8} - \frac{4}{8} = 2\frac{11}{8} - \frac{4}{8} = 2\frac{7}{8}$	116	97	260	473	....
(*c*) Added number borrowed to numerator without changing it to a fraction: $9\frac{2}{5} - 8\frac{3}{5} = 8\frac{3}{5} - 8\frac{3}{5} = 0$	177	5	9	191	....
(*d*) Borrowed unnecessarily and left improper fraction in remainder: $4\frac{2}{3} - \frac{1}{3} = 3\frac{5}{3} - \frac{1}{3} = 3\frac{4}{3}$	53	58	47	158	....
(*e*) Borrowed but disregarded fraction in minuend: $7\frac{1}{8} - \frac{5}{8} = 6\frac{8}{8} - \frac{5}{8} = 6\frac{3}{8}$	52	28	53	133	....
(*f*) Borrowed but did not change fraction in minuend to same denominator: $3\frac{1}{4} - \frac{3}{8} = 2\frac{5}{8} - \frac{3}{8} = 2\frac{2}{8}$	12	10	17	39	....
(*g*) Considered 1 borrowed where no borrowing was involved: $3\frac{3}{4} - \frac{1}{4} = 2\frac{2}{4}$	16	21	2	39	....
(*h*) Borrowed as much as was needed to make numerator in minuend larger than numerator in subtrahend: $7\frac{1}{8} - \frac{1}{4} = 5\frac{3}{8} - \frac{2}{8} = 5\frac{1}{8}$	10	0	18	28	....
2. Used wrong process	774	414	333	1,521	20.3
(*a*) Addition: $\frac{5}{8} - \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$	503	267	224	994	....
(*b*) Multiplication: $\frac{1}{4} - \frac{2}{4} = \frac{2}{16} = \frac{1}{8}$	...	2	...	2	....
(*c*) Subtracted whole numbers and added fractions: $10\frac{1}{4} - 3\frac{2}{4} = 7\frac{3}{4}$	153	87	46	286	....
(*d*) Subtracted fractions and added whole numbers: $7\frac{3}{5} - 1\frac{1}{10} = 8\frac{5}{10} = 8\frac{1}{2}$	106	48	55	209	....
(*e*) Added fractions and disregarded whole numbers: $9\frac{3}{6} - 8\frac{4}{6} = \frac{7}{6} = 1\frac{1}{6}$	6	5	4	15	....
(*f*) Added whole numbers and disregarded fractions: $3\frac{2}{6} - 1\frac{3}{12} = 4$	6	5	4	15	....
3. Difficulty in reducing fractions to lowest terms	625	311	158	1,094	14.6
(*a*) Did not reduce fraction: $7\frac{1}{5} - 3\frac{3}{6} = 3\frac{21}{30}$	581	248	118	947	....
(*b*) Divided numerator and denominator by different numbers: $8\frac{7}{12} - \frac{2}{6} = 3\frac{8}{24} = 3\frac{2}{3}$	28	44	11	83	....
(*c*) Divided denominator by numerator: $\frac{1}{2} - \frac{2}{8} = \frac{2}{8} = 4$	16	19	29	64	....

TABLE 9 (*Continued*)

	Grade V A	Grade VI B	Grade VI A	Total	Per Cent
4. Lack of comprehension of process involved	497	330	267	1,094	14.6
(*a*) Subtracted numerators and multiplied denominators: $\frac{5}{8} - \frac{1}{8} = \frac{4}{64} = \frac{1}{16}$	25	43	31	99	....
(*b*) Added numerators and subtracted denominators: $\frac{2}{4} - \frac{1}{2} = \frac{3}{2} = 1\frac{1}{2}$	2	4	3	9	....
(*c*) Subtracted numerators and added denominators: $\frac{3}{5} - \frac{2}{3} = \frac{1}{8}$	10	12	6	28	....
(*d*) Multiplied numerators and subtracted denominators: $\frac{4}{6} - \frac{3}{4} = \frac{12}{2} = 6$	...	4	..	4	....
(*e*) Multiplied numerators and added denominators: $\frac{1}{2} - \frac{2}{8} = \frac{2}{10} = \frac{1}{5}$	28	5	11	44	....
(*f*) Subtracted numerators and denominators: $\frac{2}{4} - \frac{1}{2} = \frac{1}{2}$	...	30	7	37	....
(*g*) Called common denominator answer where remainder was zero: $\frac{2}{4} - \frac{1}{2} = 4$	2	2	2	6	....
(*h*) In subtracting two equal fractions, expressed remainder by same fraction: $4\frac{2}{3} - \frac{4}{6} = 4\frac{4}{6} - \frac{4}{6} = 4\frac{4}{6} = 4\frac{2}{3}$	16	8	12	36	....
(*i*) In subtracting two equal fractions, called the remainder 1: $\frac{2}{4} - \frac{1}{2} = \frac{2}{4} - \frac{2}{4} = 1$	12	15	14	41	....
(*j*) In subtracting mixed numbers with equal fractions, placed common denominator under difference between whole numbers: $7\frac{1}{4} - 4\frac{1}{4} = \frac{3}{4}$	...	6	5	11	....
(*k*) Subtracted fraction in minuend from fraction in subtrahend: $2\frac{1}{3} - \frac{2}{3} = 2\frac{1}{3}$	62	29	23	114	....
(*l*) Added the two numerators and from that sum subtracted numerator in subtrahend: $7\frac{1}{8} - \frac{5}{8} = 7\frac{6}{8} - \frac{5}{8} = 7\frac{1}{8}$	5	5	26	36	....
(*m*) Added denominators and numerators and used the sum as numerator in minuend: $10\frac{1}{4} - 3\frac{3}{4} = 10\frac{5}{4} - 3\frac{3}{4} = 7\frac{2}{4} = 9\frac{1}{4}$	8	...	2	10	....
(*n*) In subtracting mixed number from whole number, subtracted whole numbers and placed same fraction in result: $3 - 1\frac{2}{3} = 2\frac{2}{3}$	270	164	115	549	....

TABLE 9 (*Continued*)

	Grade V A	Grade VI B	Grade VI A	Total	Per Cent
4. Lack of comprehension of process involved—*Continued*					
(*o*) When numerator in minuend was smaller than numerator in subtrahend, called remainder zero: $9\frac{3}{8} - 8\frac{4}{8} = 1\frac{0}{8}$	56	1	8	65	....
(*p*) Did not express denominator: $\frac{5}{8} - \frac{1}{8} = 4$	1	2	2	5	....
5. Difficulty in changing fractions to common denominator	288	147	191	626	8.3
(*a*) Subtracted without changing fractions to common denominator and used one of the given denominators for denominator in result: $4\frac{5}{8} - \frac{1}{4} = 4\frac{4}{8}$	229	110	160	499	....
(*b*) Changed fractions to wrong denominator: $3\frac{3}{4} - 2\frac{2}{3} = 3\frac{10}{12} - 2\frac{6}{12} = 1\frac{4}{12} = 1\frac{1}{3}$	33	27	8	68	....
(*c*) Disregarded numerator being more than 1: $7\frac{1}{5} - 3\frac{1}{2} = 7\frac{2}{10} - 3\frac{5}{10} = 4\frac{3}{10}$	13	3	17	33	....
(*d*) Changed fraction in minuend only to common denominator: $\frac{5}{6} - \frac{3}{4} = \frac{10}{12} - \frac{3}{4} = \frac{10}{16} = \frac{5}{8}$	13	7	6	26	....
6. Computation errors	230	216	168	614	8.2
(*a*) Unknown: $4\frac{3}{5} - 1\frac{2}{6} = 4\frac{18}{30} - 1\frac{10}{30} = 3\frac{8}{30}$	129	100	53	282	....
(*b*) Subtraction: $6\frac{3}{4} - 5\frac{1}{6} = 6\frac{9}{12} - 5\frac{2}{12} = 1\frac{5}{12}$	87	102	105	294	....
(*c*) Addition: $9\frac{3}{5} - 8\frac{4}{5} = 8\frac{7}{5} - 8\frac{4}{5} = \frac{3}{5}$	14	14	10	38	....
7. Omitted example (no attempt)	291	81	47	419	5.6
8. Partial operation	138	53	110	301	4.0
(*a*) Subtracted fractions but disregarded whole numbers: $4\frac{5}{8} - \frac{1}{4} = \frac{4}{8} = \frac{1}{2}$	56	46	47	149	....
(*b*) Subtracted whole numbers but disregarded fractions: $3\frac{5}{8} - 1\frac{1}{2} = 2$	82	7	63	152	....
9. Errors in copying: $3\frac{5}{6} - 1\frac{3}{12} = 3\frac{10}{12} - 1\frac{2}{12} = 2\frac{8}{12} = 2\frac{2}{3}$	1	12	1	14	0.2
Total	3,613	1,978	1,920	7,511	100.1

There are nine groups of faults in subtraction of fractions, which contain a much greater variety of errors

than was found in addition of fractions. The total number of errors is also somewhat greater than for addition. The faults are common to all grades tested.

The group containing the largest number of errors is that concerned with difficulty in borrowing, 24.3 per cent of the total. In this group the three chief faults are illustrated by the following examples:

$$(a)\ 10\tfrac{1}{4} - 3\tfrac{2}{4} = 7\tfrac{3}{4}$$

The pupil disregarded having borrowed.

$$(b)\ 3\tfrac{1}{8} - \tfrac{6}{8} = 2\tfrac{11}{8} - \tfrac{6}{8} = 2\tfrac{5}{8}$$

The pupil prefixed the borrowed number to the numerator of the fraction.

$$(c)\ 9\tfrac{3}{5} - 8\tfrac{4}{5} = 8\tfrac{4}{5} - 8\tfrac{4}{5} = 0$$

The pupil added the borrowed number (1) to the 3, making the numerator 4. This is a variation of type (*b*).

An examination of the data concerning the errors in this group shows other important types of faults. Clearly, special attention must be given to the steps involved in examples in fractions in which there is borrowing. A special set of exercises such as the following will help to locate the specific nature of the difficulty and the step in the development where it becomes evident.

Special Exercise in Borrowing

Write the missing numerators in sets (*a*) to (*f*).

(*a*)	$1 = \frac{}{2}$	$1 = \frac{}{4}$	$1 = \frac{}{6}$	$1 = \frac{}{3}$
(*b*)	$2 = 1\frac{}{2}$	$3 = 2\frac{}{4}$	$4 = 3\frac{}{3}$	$5 = 4\frac{}{5}$
(*c*)	$1\frac{1}{4} = \frac{}{4}$	$1\frac{1}{6} = \frac{}{6}$	$1\frac{3}{8} = \frac{}{8}$	$1\frac{5}{6} = \frac{}{6}$

(*d*) $2\frac{1}{4} = 1\frac{\ }{4}$ $3\frac{1}{5} = 2\frac{\ }{5}$ $7\frac{5}{8} = 6\frac{\ }{8}$ $9\frac{2}{3} = 8\frac{\ }{3}$

(*e*) $1\frac{1}{2} = \frac{\ }{4}$ $1\frac{2}{3} = \frac{\ }{6}$ $1\frac{3}{4} = \frac{\ }{8}$ $1\frac{2}{5} = \frac{\ }{10}$

(*f*) $6\frac{1}{2} = 5\frac{\ }{4}$ $7\frac{1}{3} = 6\frac{\ }{6}$ $4\frac{3}{4} = 3\frac{\ }{12}$ $5\frac{3}{5} = 4\frac{\ }{10}$

(*g*) Which of the following are incorrect?

1. $1\frac{1}{2} = \frac{3}{2}$
2. $4 = 3\frac{2}{2}$
3. $7 = 5\frac{6}{6}$
4. $1\frac{1}{3} = \frac{8}{6}$
5. $2 = 1\frac{4}{3}$
6. $5\frac{3}{4} = 4\frac{7}{4}$
7. $6\frac{1}{2} = 5\frac{6}{4}$
8. $7\frac{1}{4} = 6\frac{5}{4}$
9. $8\frac{2}{3} = 7\frac{10}{6}$
10. $8\frac{1}{2} = 7\frac{3}{2}$
11. $1\frac{2}{3} = \frac{10}{6}$
12. $6\frac{3}{5} = 5\frac{16}{10}$

(*h*) Work these examples:

1. $\begin{array}{r} 1 \\ -\ \frac{1}{2} \\ \hline \end{array}$
2. $\begin{array}{r} 2 \\ -\ \frac{1}{4} \\ \hline \end{array}$
3. $\begin{array}{r} 3 \\ -1\frac{1}{2} \\ \hline \end{array}$
4. $\begin{array}{r} 1\frac{1}{4} \\ -\ \frac{3}{4} \\ \hline \end{array}$
5. $\begin{array}{r} 3\frac{1}{4} \\ -1\frac{3}{4} \\ \hline \end{array}$
6. $\begin{array}{r} 7\frac{1}{2} \\ -2\frac{3}{4} \\ \hline \end{array}$

The next largest group of faults in subtraction of fractions was due to using the wrong process in whole or in part. In this group there are errors such as $\frac{5}{8} - \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$. The errors are probably due to lapses in attention. On the other hand there are faulty procedures which clearly indicate lack of control over the subtraction process. This is probably due to interference of some sort with work in the other processes. Confusion seems to be most marked due to interference of skills previously learned in addition.

Another large group of faults consists of procedures which clearly indicate complete lack of comprehension of the subtraction process. In this group the most common faults are illustrated by the following types:

$$(a)\ 3 - 1\tfrac{2}{3} = 2\tfrac{2}{3}$$

The pupil subtracted the integers and merely wrote the fraction in the subtrahend in the answer, thereby showing inability to borrow.

$$(b)\ 2\tfrac{1}{3} - \tfrac{3}{6} = 2\tfrac{1}{6}$$

The pupil subtracted the smaller fraction in the minuend from the larger fraction in the subtrahend, instead of borrowing; a fault similar to one frequently found in subtraction of whole numbers.

$$(c)\ \tfrac{5}{8} - \tfrac{1}{8} = \tfrac{4}{64} = \tfrac{1}{16}$$

Subtracted numerators and multiplied denominators.

$$(d)\ 9\tfrac{3}{5} - 8\tfrac{4}{5} = 1\tfrac{0}{5}$$

When the numerator in the minuend was smaller than in the subtrahend, the pupil called the difference 0, and did not borrow.

Numerous other miscellaneous procedures showing lack of comprehension of the process involved are included in the table. As can be seen from the above illustrations, one of the chief sources of difficulty in this group is inability to work examples in which borrowing is involved. When this source is added to the faulty procedures included in the first group, the necessity for special stress on the borrowing process involved in subtraction of fractions is obvious.

A special study was made to determine the relative difficulty of the various types of examples in subtraction of fractions. The per cents of error by a group of 167 sixth-grade pupils on some of the important types included in the test are given in Table 10. This table contains data on the sixteen most difficult and the sixteen easiest types of examples for this group of pupils.

TABLE 10

RELATIVE DIFFICULTY OF IMPORTANT TYPES OF EXAMPLES IN SUBTRACTION OF FRACTIONS

THE SIXTEEN MOST DIFFICULT TYPES

No.	Type	Per Cent of Error	No.	Type	Per Cent of Error
1	$3\frac{1}{6} - \frac{2}{3}$	40	9	$3\frac{1}{4} - 2\frac{5}{8}$	33
2	$4\frac{1}{4} - \frac{5}{8}$	38	10	$4\frac{5}{6} - 1\frac{2}{5}$	32
3	$4\frac{1}{8} - 1\frac{7}{8}$	34	11	$6\frac{1}{4} - 5\frac{2}{3}$	32
4	$7\frac{1}{2} - 2\frac{3}{4}$	34	12	$\frac{1}{4} - \frac{2}{8}$	32
5	$8\frac{1}{2} - \frac{2}{3}$	34	13	$3\frac{1}{2} - 1\frac{2}{3}$	31
6	$1\frac{1}{2} - \frac{5}{6}$	34	14	$1\frac{1}{10} - \frac{4}{5}$	31
7	$7\frac{1}{8} - \frac{5}{8}$	34	15	$9\frac{1}{5} - 8\frac{4}{5}$	31
8	$9\frac{1}{10} - 8\frac{1}{2}$	33	16	$6\frac{1}{3} - \frac{2}{3}$	31

THE SIXTEEN LEAST DIFFICULT TYPES

No.	Type	Per Cent of Error	No.	Type	Per Cent of Error
1	$\frac{2}{3} - \frac{1}{3}$	7	9	$4\frac{3}{5} - 2\frac{1}{5}$	14
2	$\frac{5}{4} - \frac{3}{4}$	7	10	$9 - \frac{1}{2}$	16
3	$\frac{8}{8} - \frac{3}{8}$	7	11	$2\frac{1}{2} - 2\frac{1}{2}$	16
4	$5\frac{1}{4} - 3$	8	12	$\frac{1}{2} - \frac{1}{4}$	17
5	$\frac{5}{8} - \frac{1}{8}$	10	13	$3\frac{3}{4} - 1\frac{1}{2}$	17
6	$1\frac{5}{8} - 1\frac{1}{8}$	10	14	$\frac{1}{3} - \frac{1}{4}$	17
7	$10\frac{1}{2} - 10$	13	15	$7\frac{7}{8} - 2\frac{1}{8}$	17
8	$\frac{1}{2} - \frac{1}{2}$	14	16	$4\frac{2}{3} - \frac{1}{3}$	17

An analysis of the data in Table 10 shows that 15 of the 16 most difficult types of examples involve borrowing. The only exception is the difficult type, $4\frac{5}{6}-1\frac{2}{5}$. Only one of the 16 easiest types of examples involves any borrowing difficulty. These data substantiate the results of the diagnostic study given in Table 9, which show that

borrowing presents one of the major difficulties in subtraction of fractions.

As was the case in addition of fractions, it was found that weakness in reduction of fractions and in changing fractions to a common denominator were the causes of many errors in subtraction of fractions. There were fewer errors due to faulty computation in subtraction than in addition.

(c) **Multiplication of fractions.** The chief faults in multiplication of fractions are given in Table 11.

TABLE 11

ANALYSIS OF ERRORS IN THE MULTIPLICATION OF FRACTIONS

	Grade VI B	Grade VI A	Total	Per Cent
1. Computation errors	409	303	712	28.7
(a) Division: $\frac{3}{8} \times \frac{5}{12} = \frac{1}{8}$	196	118	314	....
(b) Multiplication: $\frac{5}{6} \times 14 = \frac{59}{6} = 9\frac{5}{6}$	172	154	326	....
(c) Unknown	41	31	72	....
2. Lack of comprehension of process involved	165	264	429	17.3
(a) Inverted multiplicand: $6 \times 2\frac{1}{3} = 6 \times \frac{3}{7} = \frac{18}{7} = 2\frac{4}{7}$	53	83	136	....
(b) Inverted multiplier: $\frac{1}{4} \times 8 = \frac{4}{1} \times 8 = 32$	4	17	21	....
(c) Inverted product: $8 \times \frac{1}{6} = \frac{8}{6} = \frac{6}{8} = \frac{3}{4}$	2	9	11	....
(d) Did not express denominator in product: $3\frac{3}{4} \times 3\frac{1}{3} = \frac{15}{4} \times \frac{10}{3} = 25$	24	21	45	....
(e) Added numerators and multiplied denominators: $\frac{1}{4} \times \frac{5}{6} = \frac{6}{24}$	26	21	47	....
(f) Multiplied numerators and added denominators: $\frac{1}{4} \times 2\frac{1}{2} = \frac{1}{4} \times \frac{5}{2} = \frac{5}{6}$	16	53	69	....
(g) In multiplying whole number by fraction, multiplied whole number by denominator and added numerator to product: $\frac{1}{4} \times 8 = \frac{33}{4} = 8\frac{1}{4}$	40	60	100	....
3. Difficulty in reducing fractions to lowest terms	219	209	428	17.3
(a) Did not reduce fraction: $3 \times \frac{4}{15} = \frac{12}{15}$	142	140	282	....
(b) Divided denominator by numerator: $\frac{5}{8} \times \frac{4}{15} = \frac{5}{20} = 4$	58	49	107	....
(c) Divided numerator and denominator by different numbers: $\frac{4}{9} \times \frac{15}{4} = \frac{5}{3} = 1\frac{2}{3}$	19	20	39	....

TABLE 11 (*Continued*)

	Grade VI B	Grade VI A	Total	Per Cent
4. Omitted example (no attempt)............	151	129	280	11.3
5. Failure to change improper fractions to mixed numbers: $4 \times \frac{2}{3} = \frac{8}{3}$............	99	119	218	8.8
6. Errors in copying: $\frac{3}{8} \times 6\frac{1}{5} = \frac{\cancel{3}}{4} \times \frac{17}{\cancel{5}} = \frac{51}{20} = 2\frac{11}{20}$............	39	48	87	3.5
7. Difficulty in changing mixed numbers to improper fractions: $6 \times 2\frac{1}{3} = 6 \times \frac{3}{3} = \frac{18}{3} = 6$............	39	31	70	2.8
8. Difficulty in cancellation—canceled within numerators: $3\frac{3}{4} \times 3\frac{1}{5} = \frac{\cancel{15}^{3}}{4} \times \frac{\cancel{16}^{2}}{5} = \frac{6}{12} = \frac{1}{2}$............	17	22	39	1.6
9. Difficulty unknown: $\frac{1}{4} \times \frac{1}{6} = 46$............	99	115	214	8.6
Total............	1,237	1,240	2,477	99.9

The largest source of error in multiplication of fractions—28.7 per cent of the total—was faulty computation. This occurred chiefly in multiplication or division of the terms of the fractions. Many errors were due to lack of comprehension of the process involved, often because of confusion with the division procedure, which resulted in the inversion of one of the factors; for example,

$$6 \times 2\frac{1}{3} = 6 \times \frac{3}{7} = \frac{18}{7} = 2\frac{4}{7}$$

Sometimes the multiplier was inverted; as,

$$\frac{1}{4} \times 8 = 4 \times 8 = 32$$

Other sources of error due to this fault were the combination in some way of addition and multiplication.

Difficulty in reduction of proper and improper fractions to lowest terms and reducing mixed numbers to improper fractions caused many errors. Little difficulty in cancellation was discovered.

Relatively speaking, the process of multiplication pre-

sents fewer serious difficulties than any of the other processes, the three chief faults being inaccuracy in computation, confusion with the division procedure, and reduction of fractions and mixed numbers.

(d) **Division of fractions.** The work in division of fractions resulted in a larger number of errors by sixth-grade pupils than were found for any of the other processes. The number was practically double the total in multiplication. This may be due to the fact that division is the last of the processes in fractions that is taught and that, therefore, the amount of practice had been less than on the other processes. The analysis of errors in division is given in Table 12.

TABLE 12

ANALYSIS OF ERRORS IN THE DIVISION OF FRACTIONS

	Grade VI B	Grade VI A	Total	Per Cent
1. Used wrong process—multiplication: $1\frac{3}{8} \div 1\frac{2}{3} = \frac{11}{8} \times \frac{5}{3} = \frac{55}{24} = 2\frac{7}{24}$	723	795	1,518	31.1
2. Computation errors	365	309	674	13.8
(a) Division: $3\frac{3}{8} \div 1\frac{1}{4} = \frac{27}{8} \times \frac{4}{7} = \frac{27}{14} = 1\frac{13}{14}$	255	223	478	....
(b) Multiplication: $1\frac{1}{6} \div 3\frac{1}{2} = \frac{7}{6} \times \frac{2}{7} = \frac{14}{36} = \frac{2}{6}$	98	78	176	....
(c) Unknown: $3\frac{1}{3} \div 1\frac{3}{4} = \frac{10}{3} \times \frac{4}{7} = 3\frac{1}{21}$	12	8	20	....
3. Lack of comprehension of process involved	219	371	590	12.1
(a) Inverted dividend: $1\frac{1}{6} \div 3\frac{1}{2} = \frac{6}{7} \times \frac{7}{2} = \frac{42}{14} = 2\frac{14}{14}$	43	201	244	....
(b) Inverted both dividend and divisor: $1\frac{3}{8} \div 1\frac{2}{3} = \frac{8}{11} \times \frac{5}{3} = \frac{40}{33}$	50	38	88	....
(c) Added denominators and multiplied numerators: $1\frac{3}{8} \div 1\frac{2}{3} = \frac{11}{8} \times \frac{5}{6} = \frac{33}{13} = 2\frac{7}{13}$	43	52	95	....
(d) Added numerators and multiplied denominators: $1\frac{1}{6} \div 3\frac{1}{3} = \frac{8}{6} \times \frac{3}{7} = \frac{8}{35}$	27	18	45	....
(e) Disregarded denominator in quotient: $3\frac{1}{8} \div 1\frac{1}{4} = \frac{25}{8} \times \frac{4}{5} = 5$	47	45	92	....
(f) Disregarded numerator in quotient: $\frac{1}{6} \div \frac{1}{2} = \frac{1}{6} \times \frac{2}{1} = 3$	9	17	26	....

TABLE 12 (*Continued*)

	Grade VI B	Grade VI A	Total	Per Cent
4. Difficulty in reducing fractions to lowest terms	259	177	436	8.9
(*a*) Did not reduce fraction: $1\frac{1}{3} \div 3\frac{1}{3} = \frac{4}{3} \times \frac{3}{10} = \frac{4}{10}$	223	150	373	....
(*b*) Divided denominator by numerator: $1\frac{3}{8} \div 1\frac{2}{3} = \frac{11}{8} \times \frac{8}{5} = \frac{40}{33} = 1\frac{7}{33}$	36	27	63	....
5. Difficulty in changing mixed numbers to improper fractions: $3\frac{1}{3} \div 1\frac{3}{4} = \frac{10}{3} \times \frac{4}{12} = \frac{20}{18} = 1\frac{1}{9}$	220	201	421	8.6
6. Omitted example (no attempt)	192	214	406	8.3
7. Failure to change improper fractions to mixed numbers: $3\frac{1}{8} \div 1\frac{1}{4} = \frac{25}{8} \times \frac{4}{5} = \frac{5}{2}$	223	126	349	7.2
8. Errors in copying: $1\frac{1}{2} \div \frac{1}{5} = \frac{3}{2} \times \frac{5}{1} = \frac{15}{2} = 7\frac{1}{2}$	61	52	113	2.3
9. Difficulty in cancellation	11	63	74	1.5
(*a*) Canceled within denominators: $\frac{5}{8} \div 4 = \frac{5}{8} \times \frac{1}{4} = \frac{5}{2}$	2	9	11	....
(*b*) Canceled within numerators: $1\frac{1}{5} \div 3\frac{1}{3} = \frac{6}{5} \times \frac{3}{10} = \frac{3}{50}$	5	14	19	....
(*c*) In complete cancellation called quotient zero: $4\frac{1}{2} \div 4\frac{1}{2} = \frac{9}{2} \times \frac{2}{9} = 0$	4	40	44	....
10. Difficulty unknown: $1\frac{1}{3} \div 3\frac{1}{6} = \frac{1}{6}$	75	219	294	6.0
Total	2,348	2,527	4,875	99.8

Almost one-third of the errors in division of fractions, 31.1 per cent of the total, were due to the use of the multiplication instead of the division procedure; for example,

$$1\frac{3}{8} \div 1\frac{2}{3} = \frac{11}{8} \times \frac{5}{3} = \frac{55}{24} = 2\frac{7}{24}$$

The pupils failed to invert the divisor. Faulty computations caused 13.8 per cent of the errors. Almost as many errors were due to lack of knowledge of basic processes involved in division. Some pupils inverted the dividend; others inverted both dividend and divisor, although this fault was found less frequently than the inversion of the dividend. Other types of errors com-

ned addition of the terms or multiplication of terms in faulty ways.

As was found in the other processes, failure to reduce proper and improper fractions in answers to lowest terms constituted a major source of error. Likewise, many errors were due to difficulty in changing mixed numbers to improper fractions. A small amount of error was found due to difficulty in cancellation. Sometimes pupils canceled within the numerators; sometimes within the denominators. For the type of example involving complete cancellation,

$$4\tfrac{1}{2} \div 4\tfrac{1}{2} = \frac{\cancel{9}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{9}} = 1,$$

the answer was often given as zero.

(e) **Summary of the investigation of faults in fractions.** 1. An error in computation is one of the major difficulties in work with fractions. The percentage of errors due to computation being 13.8 in addition, 8.2 in subtraction, 28.7 in multiplication, and 13.8 in division.

2. The major causes of errors in each process are as follows:

Addition:	Per Cent
(*a*) Lack of comprehension of process involved	20.2
(*b*) Difficulty in reducing fractions to lowest terms	17.5
(*c*) Difficulty with improper fractions	17.1
(*d*) Computation errors	13.8
Total	68.6[1]

Subtraction:	
(*a*) Difficulty in borrowing	24.3
(*b*) Used wrong process	20.3
(*c*) Difficulty in reducing fractions to lowest terms	14.6
(*d*) Lack of comprehension of process involved	14.6

[1] Percentage of difficulties unknown, 20.4.

	Per Cent
(*e*) Difficulty in changing fractions to common denominator	8.3
(*f*) Computation errors	8.2
Total	90.3
Multiplication:	
(*a*) Computation errors	28.7
(*b*) Lack of comprehension of process involved	17.3
(*c*) Difficulty in reducing fractions to lowest terms	17.3
(*d*) Omitted example (probably lack of comprehension)	11.3
(*e*) Failure to change improper fractions to mixed numbers	8.8
(*f*) Difficulty in changing mixed numbers to improper fractions	2.8
Total	86.2
Division:	
(*a*) Used wrong process	31.1
(*b*) Computation errors	13.8
(*c*) Lack of comprehension of process involved	12.1
(*d*) Difficulty in reducing fractions to lowest terms	8.9
(*e*) Difficulty in changing mixed numbers to improper fractions	8.6
(*f*) Omitted example (lack of comprehension of process)	8.3
(*g*) Failure to change improper fractions to mixed numbers	7.2
Total	90.0

3. The major difficulties in all processes are (*a*) lack of comprehension of process involved, (*b*) difficulty in reducing fractions to lowest terms, and (*c*) difficulty in changing improper fractions to whole or mixed numbers.

4. Difficulty in changing fractions to a common denominator is a cause of relatively few errors in addition of fractions but a significant cause of errors in subtraction.

5. Changing mixed numbers to improper fractions is a major cause of difficulty in both multiplication and division of fractions.

6. There is clear evidence that the kinds of errors that have been analyzed exist in each of the grades studied. Supplementary investigations show that they are also found in large numbers in the upper grades.

7. The analysis of errors here reported is based on a detailed study of the written work of pupils. It should be supplemented by an individual study of the work of pupils who have special difficulties. Such a study should be similar to the studies of Buswell and John, and the writer, in analyzing the work of pupils in the fundamental processes. The many peculiar types of errors listed in the tables clearly show the need of special attention to individual difficulties. This is especially true in the addition of fractions. It was not possible to analyze the causes of the errors in 20.4 per cent of the cases by an examination of the written work alone.

Problems for Study, Reports, and Discussion

1. Make an analysis of the unit skills in the examples in one of the diagnostic tests in fractions included in this chapter.

2. Prepare a set of examples that duplicates one of the diagnostic tests included in this chapter.

3. Arrange the types of examples in one of the diagnostic tests in the order of their difficulty. Use the facts presented in the tables in this chapter.

4. How should the facts regarding difficulty of types of examples help the teacher?

5. What are the five most common types of faults in addition of fractions? in subtraction? in multiplication? in division?

6. Give a diagnostic test in a process in fractions to a

sixth-grade class. Analyze the work to discover how frequently each type of example was worked incorrectly and determine the causes of each error.

7. What types of difficulties could be overcome by means of specific learning units designed to eliminate or to prevent them? Examine textbooks to see if special provision is made for such learning units.

8. How do you account for the extreme difficulty of types, involving zero difficulties in subtraction of fractions?

9. Explain the cause of difficulty in solving this multiplication example: $\frac{1}{5} \times \frac{4}{5} = \frac{4}{5}$.

10. Would you teach pupils to manipulate such fractions as sevenths, elevenths, and thirty-firsts? Why?

11. What are the arguments against the teaching of cancellation?

12. Which are being used more widely at the present time, common or decimal fractions? Which do you think is likely to be used more extensively in the future?

13. Apply Dr. Osburn's technique in the analysis of the steps involved in solving the example: $7\frac{1}{8} + 8\frac{9}{10}$.

14. Why would it be desirable to individualize instruction in fractions?

15. Prepare a list of ten problems illustrating the use of fractions in every day life.

Selected References

1. Brueckner, L. J., *Diagnostic Test in Fractions,* Manual (Minneapolis, Minnesota: Educational Test Bureau, 1926).

2. Knight, F. B., "Report on Organization of Drill," *Third Yearbook of the Department of Superintendence* (Washington, D. C., 1925), pp. 63–91.

3. Morton, R. L., *Teaching Arithmetic in the Intermediate Grades* (Newark, N. J.: Silver Burdett and Company, 1927).

4. Osburn, W. F., *Corrective Arithmetic,* Vol. 1 (Boston: Houghton Mifflin Company, 1924).

5. *Twenty-Ninth Yearbook of the National Society for the Study of Education,* Part I (Bloomington, Illinois: Public School Publishing Company, 1930), pp. 207–12, and Part II, Chapter 5.

CHAPTER VI

DIAGNOSIS OF DIFFICULTIES IN DECIMALS AND PER CENT

This chapter contains the results of a study of the errors made by pupils in Grades VI, VII, and VIII in analytical diagnostic tests[1] on the four processes in decimals. The basis of the analysis was a study of the written work of more than three hundred pupils in these grades in four different schools in Minneapolis. The investigation sought to ascertain the types of examples in each process in decimals found most difficult, and to discover the causes of the errors made.

1. DIAGNOSTIC TESTS IN DECIMALS.

The tests used were as follows: Part 1 consists of examples in reading, writing, and converting decimals, and in comprehending the value of decimals. Part 2 consists of diagnostic tests in each of the four processes. Each test contains a wide variety of types of examples involving different skills, or presenting special kinds of difficulties. The results of these tests made it possible to sample the entire series of skills in each process in decimals. This is a much more satisfactory procedure than to use tests consisting of only a few types. Such

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[1] Similar diagnostic tests in decimals may be found in L. J. Brueckner, C. J. Anderson, G. O. Banting, and E. Merton, *Diagnostic Tests and Practice Exercises*, Grades 3, 4, 5, 6, 7, and 8 (Philadelphia: The John C. Winston Company, 1929) and in *The Triangle Arithmetics*, also published by The John C. Winston Company.

examples may contain only a small number of the specific elements and consequently may omit the very types that are the sources of the largest numbers of errors. Pupils were given as much time as they needed to work the examples in the tests.

BRUECKNER DIAGNOSTIC TEST IN DECIMALS[1]

NAME.. GRADE................ AGE..........

SCHOOL.. DATE..........................

I. Below are numbers written as words. Write each in decimal form.
   a. Five tenths..
   b. Four hundredths..
   c. Sixteen hundredths...
   d. Three thousandths..
   e. Forty six thousandths..
   f. One hundred seven thousandths...............................
   g. Forty nine and eighty four hundredths.....................

II. Write the numbers below as words.
   a. .65 ....
   b. 1.4 ....
   c. 8.75 ....
   d. .146 ....
   e. 1.09 ....
   f. 20.056 ....

III. Arrange the following numbers in order of their size. Write the number of greatest value first.

		Write here
a.	23.3	1. ....
b.	2.33	2. ....
c.	233.0	3. ....
d.	2.303	4. ....
e.	200.33	5. ....

IV. Express as decimals: (a) $\frac{7}{8}$ .... (b) $1\frac{1}{4}$ ....
(c) $\frac{9}{100}$ .... (d) $7\frac{5}{8}$ ....

[1] Published by Educational Test Bureau, Minneapolis, Minnesota.

V. Express as common fractions: (a) .5 .... (b) .75 ....
(c) .09 .... (d) .149 ....

Addition

Can you find the following sums correctly?

1.	2.	3.	4.	5.
.4	.3	.16	.28	.02
.1	.5	.25	.43	.03
.3	.8	.37	.95	.04

6.	7.	8.	9.
.05	1.06	1.75	2.75
.09	2.08	2.125	4.
.08	3.04	3.8	16.375

Work No. 10 here

Work No. 11 here

10. Find the sum of 9.65 + 8.375 + 6.4
11. Find the sum of .8 + 3 + .125
12. $\frac{1}{4}$ + .25 = ...

Subtraction

Can you do these subtraction examples correctly?

1.	2.	3.	4.	5.
.8	.5	.16	.38	.43
.3	.5	.04	.15	.41

6.	7.	8.	9.	10.
.375	.7	.9	.4	.6
.269	.35	.275	.375	.004

11.	12.	13.	14.	15.
9.6	18.5	27.08	9.3	18.2
3.4	4.6	15.17	6.25	1.625

Work No. 16 here

Work No. 17 here

16. Subtract 3.825 from 20
17. Subtract .5 from .75

Multiplication

Can you do these multiplication examples correctly?

1. $4 \times .2 =$ .... 2. $4 \times .02 =$ .... 3. $4 \times .002 =$ ......

4. $5 \times .3 =$ .... 5. $6 \times .04 =$ .... 6. $7 \times .008 =$ ......

7. $8 \times .5 =$ .... 8. $6 \times .05 =$ .... 9. $4 \times .005 =$ ......

10. $\begin{array}{r} 2.6 \\ \underline{4} \end{array}$ 11. $\begin{array}{r} 3.5 \\ \underline{8} \end{array}$ 12. $\begin{array}{r} 3.28 \\ \underline{4} \end{array}$ 13. $\begin{array}{r} 4.647 \\ \underline{5} \end{array}$

14. $\begin{array}{r} 85 \\ \underline{.4} \end{array}$ 15. $\begin{array}{r} 20 \\ \underline{.6} \end{array}$ 16. $\begin{array}{r} 32 \\ \underline{1.4} \end{array}$ 17. $\begin{array}{r} 2.5 \\ \underline{48} \end{array}$

18. $.4 \times .2 =$ .... 19. $.5 \times .03 =$ .... 20. $.8 \times .25 =$ ......

21. $\begin{array}{r} 7.8 \\ \underline{6.4} \end{array}$ 22. $\begin{array}{r} 6.5 \\ \underline{4.8} \end{array}$ 23. $\begin{array}{r} 18.4 \\ \underline{.26} \end{array}$ 24. $\begin{array}{r} 8.04 \\ \underline{.03} \end{array}$

25. $10 \times 8.5 =$ .... 26. $10 \times .96 =$ .... 27. $100 \times .14 =$ ....

28. $100 \times 8.5 =$ .... 29. $100 \times 8.65 =$ .... 30. $200 \times 9.4 =$ ....

31. $\frac{3}{8}$ of $6.4 =$ .... 32. $.08 \times 25 \times \frac{1}{2} =$ ....

Division

Can you work all these division examples?

1. $4\overline{)\,8.4}$ 2. $3\overline{)\,7.47}$ 3. $8\overline{)\,16.896}$

4. $2\overline{)\,.8}$ 5. $4\overline{)\,.76}$ 6. $6\overline{)\,.972}$

7. $4\overline{)\,.12}$ 8. $6\overline{)\,.042}$ 9. $24\overline{)\,1.2}$

10. $8\overline{)\,4}$ 11. $8\overline{)\,6}$ 12. $25\overline{)\,2}$

13. $25\overline{)\,64}$ 14. $33\overline{)\,87}$ 15. $25\overline{)\,8.725}$

16. $.3\overline{)\,3.6}$	17. $.3\overline{)\,18.63}$	18. $.4\overline{)\,1.2}$
19. $.3\overline{)\,6}$	20. $.2\overline{)\,10}$	21. $.4\overline{)\,3}$
22. $.7\overline{)\,4}$	23. $.11\overline{)\,1.21}$	24. $.11\overline{)1.342}$
25. $.11\overline{)\,3.3}$	26. $.12\overline{)\,6}$	27. $.12\overline{)\,9}$
28. $1.25\overline{)\,6.75}$	29. $6.48\overline{)\,7.128}$	30. $.834\overline{)\,91.74}$

To illustrate the differences between the examples in addition note the following typical points:

(1) $\begin{array}{r} .4 \\ .1 \\ .3 \\ \hline .8 \end{array}$ In example 1 the sum is less than 10. The decimal point in the sum is placed directly below the decimal points in the addends.

(2) $\begin{array}{r} .3 \\ .5 \\ .8 \\ \hline 1.6 \end{array}$ In example 2 the sum of the addends is greater than 10. The chief element of difficulty involved is placing the decimal point properly.

(5) $\begin{array}{r} .02 \\ .03 \\ .04 \\ \hline .09 \end{array}$ In example 5 the sum of the hundredths column is less than 10. The element that causes difficulty here is the handling of the zeros and the placing of the decimal point.

(11) Find the sum of .8 + 3 + .125. In this example the pupil must copy the numbers, place them correctly, know how to add decimals involving blank spaces, and to place the decimal point correctly.

A similar analysis can be made for the examples in the test in subtraction.

The major problem in the construction of a diagnostic test in multiplication of decimals is to make certain that the pupil encounters the types of situations which may

present difficulty in the placement of the decimal point in the product. Note the following types of examples taken from the test in multiplication:

(1) $4 \times .2 = .8$

The product is less than 10 and the pupil must place the decimal point before the 8.

(4) $5 \times .3 = 1.5$

The product is greater than 10 and the pupil must point off only one place.

(19) $.5 \times .03 = .015$

The difficulty introduced here is the necessity of prefixing a zero to fill in the empty place.

(29) $100 \times 8.65 = 865$

The difficulty involved is the shifting of the decimal point to the right the proper number of places.

The analysis of the elements involved in these four examples in multiplication can be extended to the other examples in the test. In all there are 32 types of multiplication examples in the test.

As has been pointed out by Monroe,[1] the question of determining the major difficulties in division of decimals is largely a matter of testing the ability of the pupil to work the many different possible types of examples in that process. Each example should contain a particular combination of the elements which cause difficulty. These elements are those involving the placement of the decimal point in the quotient, the procedure to follow when there are remainders, and the method of working examples when decimals are included in both the dividend and the divisor. The diagnostic test in division

[1] W. S. Monroe, "The Ability to Place the Decimal Point in Division," *Elementary School Journal*, Vol. 18, pp. 287–98.

contained 30 different types of examples, each of which involves a different element or combination of skills. The following will illustrate the method of determining the differences between the examples:

(1)	$4\overline{)8.4}$ quotient 2.1	This is a simple example. There are no difficulties, other than the placing of the decimal point in the quotient.
(4)	$2\overline{).8}$ quotient .4	The division is simple. The quotient is a decimal.
(8)	$6\overline{).042}$ quotient .007	The new element here is the insertion of zeros to fill in empty places in the quotient.
(19)	$6 \div .3 =$ $.3\overline{)6.0}$ quotient 2 0	Here it is necessary to annex a zero to the dividend because the divisor is a decimal, tenths, and the dividend is a whole number, 6.
(29)	$6.48\overline{)7.12\ 8}$ quotient 1.1	This example involves the division of a decimal containing thousandths by a decimal containing hundredths, resulting in a special difficulty in placing the decimal point in the quotient.

2. Relative Difficulty of Examples in Decimals.

The diagnostic tests in decimals were given to pupils in grades 6, 7, and 8 in order to secure a measure of the difficulty of the various types of examples in the tests. Table 13 contains the per cents of error made by a group of 168 Minneapolis seventh-grade students on the examples contained in each of the four processes in the diagnostic tests on pages 220 to 222. The numbers of the examples in the table correspond to the numbers of the examples in the tests.

TABLE 13

Per Cents of Error by 168 Seventh-Grade Pupils

A. Addition

Example	Per Cent of Error	Example	Per Cent of Error
1	2	7	8
2	50	8	4
3	4	9	10
4	26	10	8
5	2	11	17
6	4	12	49

B. Subtraction

Example	Per Cent of Error	Example	Per Cent of Error
1	1	9	17
2	1	10	17
3	2	11	1
4	1	12	5
5	1	13	4
6	4	14	7
7	7	15	18
8	15	16	21
		17	25

C. Multiplication

Example	Per Cent of Error	Example	Per Cent of Error
1	2	17	9
2	4	18	14
3	4	19	20
4	17	20	12
5	8	21	11
6	9	22	12
7	17	23	15
8	9	24	8
9	12	25	13
10	8	26	20
11	12	27	10
12	16	28	17
13	18	29	15
14	15	30	21
15	8	31	48
16	15	32	56

D. Division

Example	Per Cent of Error	Example	Per Cent of Error
1	3	16	32
2	15	17	30
3	5	18	28
4	2	19	55
5	7	20	59
6	11	21	59
7	16	22	71
8	12	23	33
9	25	24	36
10	29	25	37
11	35	26	50
12	34	27	55
13	48	28	51
14	59	29	34
15	28	30	39

E. Summary

Process	Range	Median
Addition	2–49	8.0
Subtraction	1–25	5.0
Multiplication	2–56	14.5
Division	2–71	34.0

The data in Table 13 show that there is a large variation in the difficulty of the various types of examples in each process. This is shown by the per cents of all pupils who made errors on the examples. Part E of Table 13 shows that the variation in per cents of error on the addition test was from 2 to 49 per cent, on the subtraction test, from 1 to 25 per cent, on the multiplication test, from 2 to 56 per cent, and on the division test, from 2 to 71 per cent. This variation is practically as great as was found for types of examples in fractions. It should be pointed out that the computations involved in the examples on the tests are very simple. The cause of difficulty in solving the examples is therefore chiefly that of manipulating the decimal points.

The data that have been described show that systematic practice must be given on a wide variety of types of examples in each process in decimals. An analysis of the facts in Table 13 will aid the teacher to discover the most difficult types in each process. Similar tests should be given to every class working on the decimal processes to discover the types of examples that are causing difficulty. Carefully constructed practice exercises can be prepared in the light of the known facts regarding difficulty of types. Special practice should of course be given on the difficult types.

3. Analysis of Faults in Decimals.

After the tests had been given to the pupils, the first step was to check the examples that were incorrect and to determine the causes of the errors by an examination of the written work of the pupils. In most cases this was a relatively easy matter. It should be pointed out that the computations in the tests were kept as simple as possible so that errors due to faulty handling of decimals would be revealed rather than errors due to difficult computation. The reasons for the errors were then classified according to type.

The total number of different kinds of errors isolated was 114, the largest variety occurring in grade VIII. The kinds of errors were more numerous in multiplication and division than in addition and subtraction. They were distributed as follows: general difficulties in reading, writing, and conversion, 33; difficulties in addition, 15; difficulties in subtraction, 14; difficulties in multiplication, 26; and difficulties in division, 26.

There was no evidence that errors are at all typical by grades. Therefore, in the tables, the errors for all grades

are combined. The tables contain the classification of a total of 8,785 errors. The analysis of the errors is obvious from the statements of the errors given in the tables.[1]

(a) **Difficulties in Reading, Writing, and Converting Decimals.** What seemed to the investigator to be an astonishingly large number of difficulties was revealed by Part 1 of the test. Table 14 contains the summary of errors in this part of the test.

TABLE 14

DIFFICULTIES IN READING, WRITING, AND CONVERTING DECIMALS

	Frequency
1. Lack of comprehension of numerical values of decimals	519
2. Difficulties in expressing decimal numbers in words:	
(*a*) Errors in spelling	451
(*b*) Omission of essential words	15
(*c*) Inability to write decimal fractions in words	15
3. Difficulties in reading and writing decimals:	
(*a*) Inability to write fractions as decimals	95
(*b*) Misplacing of decimal point	88
(*c*) Inability to express mixed numbers in decimals	55
(*d*) Zero difficulties:	
(1) Placing of extra zero in answer	26
(2) Misplacing of zero	22
(*e*) Writing decimals as part common fractions and part decimal fractions	38
4. Difficulties in writing decimals as common fractions:	
(*a*) Inability to reduce fractions to lowest terms	58
(*b*) Inability to write decimals as common fractions	44
5. Lack of fundamental knowledge	261
6. Other difficulties:	
(*a*) Mathematical	0
(*b*) Non-mathematical:	
(1) No attempt	128
(2) Carelessness in reading	120

[1] The data in the following tables have previously appeared in the article by the author, "Analysis of Difficulties in Decimals," *Elementary School Journal*, Vol. 29, pp. 32–44.

	Frequency
(3) Failure to write out completely	77
(4) Failure to follow directions	64
(5) Work incomplete	22
7. Miscellaneous	77
Total	2,175

The largest number of errors, 519 in all, was made in example 3 on the test in which the pupils were asked to arrange the numbers in the order of their value. The many errors suggest that the pupils did not have a clear notion of the value of place in the decimal system. It may be that we assume that the pupil learns the value of place from work in the four processes in decimals. The results of this study show that considerably more stress needs to be given to this phase of work with decimals before we can be certain that pupils understand the place value of numbers in the decimal system.

Many errors were due to misspelling when pupils were asked to express decimals in words, the chief errors being the use of *hundreds* for *hundredths*, and *thousands* for *thousandths*. Considerable difficulty was found in reducing fractions to decimals, or decimals to fractions. In many cases these examples on the test were not even attempted, showing probably complete lack of knowledge of how to proceed. Evidently the reduction of fractions to decimals and vice versa had not been sufficiently stressed in these classes.

(b) **Difficulties in Addition and Subtraction of Decimals.** As can be seen from the data in Table 15, the two chief sources of error in addition of decimals are errors in addition and incorrect placement of the decimal point in the sum. The latter type of error occurred especially in examples 2, 4, 5, and 6 in the test. In 34 cases the

numbers were not placed correctly in examples 11 or 12. In only a few cases was the decimal point omitted.

TABLE 15

DIFFICULTIES IN THE ADDITION OF DECIMALS

	Frequency	
1. Difficulties basic to any addition:		
(a) Errors in number combinations	128	
(b) Difficulties in carrying	31	159
2. Difficulties peculiar to decimal situations:		
(a) Misplacing of decimal point	275	
(b) Difficulties in adding common fractions and decimals:		
(1) Inability to add	23	
(2) Fractions added to decimals ($.4 + \frac{1}{2} = .4\frac{1}{2}$)	20	
(c) Misplacing of whole number	34	
(d) Omission of decimal point	3	355
3. Other difficulties:		
(a) Mathematical (miscellaneous)	5	
(b) Non-mathematical:		
(1) No attempt	36	
(2) Failure to follow directions	14	
(3) Work incomplete	7	
(4) Carelessness	4	66
Total		580

TABLE 16

DIFFICULTIES IN THE SUBTRACTION OF DECIMALS

	Frequency	
1. Difficulties basic to any subtraction:		
(a) Difficulties in borrowing	221	
(b) Errors in subtraction	50	
(c) Subtraction confused with addition	45	
(d) Zero difficulties	15	
(e) Subtrahend and minuend reversed	12	343
2. Difficulties peculiar to decimal situations:		
(a) Misplacing of decimal number in subtrahend	74	
(b) Omission of decimal point	17	
(c) Misplacing of decimal point	8	99

TABLE 16 (*Continued*)

		Frequency
3. Other difficulties:		
(*a*) Mathematical (miscellaneous)	5	
(*b*) Non-mathematical:		
(1) No attempt	11	
(2) Work incomplete	4	
(3) Carelessness	3	23
Total		465

In subtraction the chief source of error was lack of control over the procedure to use when borrowing was needed, especially in examples 7, 8, 9, 10, 14, and 15. In these examples blank places appeared in the minuend. Almost half of all errors on the subtraction test were due to this difficulty. A considerable number of errors was due to errors in subtraction and to confusion of the subtraction and addition processes. The incorrect placement of the subtrahend in such examples as:

Subtract .5 from .75,

also was a major source of error due to the same difficulty listed for addition, namely, probable lack of understanding of the place value of the decimal system. In a few cases the decimal point was omitted or misplaced.

(c) **Difficulties in Multiplication of Decimals.** The results of the analysis of difficulties in the multiplication of decimals are given in Table 17:

TABLE 17

DIFFICULTIES IN THE MULTIPLICATION OF DECIMALS

		Frequency
1. Difficulties basic to any multiplication:		
(*a*) Errors in multiplication	365	
(*b*) Difficulties in carrying	38	
(*c*) Errors in addition of partial products	34	
(*d*) Multiplier or multiplicand copied as answer	17	
(*e*) Inability to multiply by zero	11	465

TABLE 17 (*Continued*)

		Frequency
2. Difficulties peculiar to decimal situations:		
(*a*) Placement of decimal point:		
(1) Misplacing of decimal point	631	
(2) Omission of decimal point	119	
(*b*) Zero difficulties:		
(1) Failure to prefix zero	87	
(2) Prefixing of unnecessary zero	38	
(3) Annexing of unnecessary zero	19	
(4) Failure to annex zero	10	
(*c*) Inability to express common fractions as decimals:		
(1) Inability to multiply decimals and fractions	62	
(2) Answers written in fraction form	52	
(*d*) Multiplied whole numbers and added decimals	13	
(*e*) Multiplied whole numbers and decimals separately	2	1,033
3. Other difficulties:		
(*a*) Mathematical:		
(1) Misplacing of zero	10	
(2) Miscellaneous	20	
(3) Added instead of multiplied	36	66
(*b*) Non-mathematical:		
(1) No attempt	168	
(2) Unknown	52	
(3) Work incomplete	24	
(4) Carelessness	6	250
Total		1,814

The data show that there were three chief sources of error in multiplication of decimals, namely:

1. Difficulties basic to any multiplication, such as errors in combinations or in carrying.
2. The placing of the decimal point in products.
3. The omission of the decimal point.

The use of zeros caused many errors, either through

failure to prefix or annex zeros when necessary, or through the prefixing or annexing of unnecessary zeros. The following illustrate typical errors:

(*a*) $4 \times .02 = .8$	Omission of a necessary zero.
(*b*) $5 \times .03 = .015$	Insertion of an unnecessary zero.
(*c*) $6 \times .36 = 216$	Omission of decimal point.
(*d*) $5 \times .3 = .15$	Misplacement of decimal point.

Many pupils failed in the example, $\frac{3}{8}$ of 6.4, or omitted it entirely, again displaying weakness in converting fractions to decimals, or being unable to work examples of this type because the type never had been presented in the development.

(d) **Difficulties in Division of Decimals.** More errors were found in division of decimals than in any of the other processes.

TABLE 18

DIFFICULTIES IN THE DIVISION OF DECIMALS

	Frequency	
1. Difficulties basic to any division:		
(*a*) Errors in division	376	
(*b*) Difficulties with trial divisor	99	
(*c*) Misplacing of remainder	83	
(*d*) Errors in multiplication	47	
(*e*) Errors in subtraction	11	616
2. Difficulties peculiar to decimal situations:		
(*a*) Placement of decimal point:		
(1) Misplacing of decimal point	1,436	
(2) Omission of decimal point	356	
(3) Decimal point used when unnecessary	83	
(*b*) Zero difficulties:		
(1) Failure to prefix zero in quotient	163	
(2) Failure to annex zero to dividend	143	
(3) Failure to annex zero to quotient	106	
(4) Placing of extra zeros in quotient	85	

TABLE 18 (*Continued*)

		Frequency
(5) Prefixing of unnecessary zeros with resulting wrong placement of decimal point	34	
(6) Misplacing of zero in quotient.........	19	
(*c*) Failure to reduce remainder to decimal.....	172	2,597
3. Other difficulties:		
(*a*) Mathematical:		
(1) Failure to know correct procedure.....	89	
(2) Miscellaneous....................	39	
(*b*) Non-mathematical:		
(1) Work incomplete....................	202	
(2) No attempt.........................	198	
(3) Carelessness.......................	10	538
Total................................		3,751

As was expected, a considerable proportion of the total number of errors was due to faulty combinations in the division process, approximately one-sixth of the total.

The largest single cause of errors was due to misplacement of the decimal point or its omission. This difficulty arose chiefly in examples in which the divisor was a decimal and the dividend a whole number, such as

$$.4\overline{)3}\,,\quad .12\overline{)9}\,,\quad .2\overline{)10}\,,\quad \text{and}\quad .7\overline{)4}$$

The use of zeros also was a source of considerable difficulty. The most common errors in this group were:

(*a*) failure to prefix necessary zeros in the quotient; as $\begin{array}{r} .6.\\ 6\overline{).036}\end{array}$

(*b*) failure to annex zero to dividend; as $\begin{array}{r} 3\\ .2\overline{)6}\end{array}$

(*c*) failure to annex zero to quotient; as $\begin{array}{r} 3\\ .2\overline{)6.0}\end{array}$

(*d*) placing of extra zeros in quotient; as $\begin{array}{r} .03\\ 4\overline{)1.2}\end{array}$

Many pupils neglected to reduce the remainder to a decimal in such examples as, $33\overline{)87.}$

Whether this was due to lack of knowledge of how to proceed or because the directions were not definite enough is not known. This point shows the desirability of interviewing pupils whose work contains certain types of errors or shortcomings, to determine the specific causes. This procedure might have cleared up the reasons why in many cases the work was incomplete or the example not attempted. A very worth-while study of this type could be made by someone interested in making a study for decimals similar to that made by Buswell and John for the fundamental processes.

(e) **Summary of the Investigation.** 1. Many pupils do not have adequate concepts of the numerical values of decimals. This can be seen in Table 14 from the 519 errors due to this difficulty. Only two other types of errors occurred as frequently, namely, misplacing the decimal point in division, and misplacing the decimal point in multiplication.

2. Many errors were due to the misspelling of the decimal written in word form; for example, "hundreds" for "hundredths."

3. Failure to place the decimal point correctly was the greatest cause of errors in addition, $.3 + .5 + .8 = .16$ being the most common type.

4. The number of errors in addition due to inaccuracy was about half as great as the number of errors due to the misplacement of the decimal point.

5. The greatest difficulties in subtraction were in borrowing and in the placement of the decimal number in the subtrahend. There were few errors due to inaccuracy.

6. The major difficulty in multiplication of decimals

was the misplacing of the decimal point or its complete omission.

7. There were many errors due to inaccuracy in multiplication.

8. The major causes of errors in division were the misplacing of the decimal point, faulty placement of zeros, omission of the decimal point, and inaccuracy.

9. The types of examples in each of the four processes in decimals worked incorrectly most frequently are as follows:

Addition

1. $\begin{array}{r} 2.75 \\ 4 \\ \underline{16.375} \end{array}$

2. Find the sum of $.8 + 3 + .125$

3. $\begin{array}{r} .28 \\ .43 \\ \underline{.95} \end{array}$

4. $.25 + \frac{1}{4} =$

5. $\begin{array}{r} .3 \\ .5 \\ \underline{.8} \end{array}$

Subtraction

1. $\begin{array}{r} 18.2 \\ \underline{1.625} \end{array}$

2. $\begin{array}{r} .4 \\ \underline{.375} \end{array}$

3. $\begin{array}{r} .6 \\ \underline{.004} \end{array}$

4. Subtract 3.825 from 20
5. Subtract .5 from .75

Multiplication

1. $\begin{array}{r} 4.647 \\ \underline{5} \end{array}$

2. $.5 \times .03 =$

3. $200 \times 9.4 =$

4. $\frac{3}{8}$ of $6.4 =$

5. $.08 \times 25 \times \frac{1}{2} =$

DIVISION

1. $33\overline{)87}$	2. $.4\overline{)3}$	3. $.12\overline{)9}$
4. $.3\overline{)6}$	5. $.2\overline{)10}$	6. $.7\overline{)4}$

4. DIAGNOSIS OF DIFFICULTIES IN PER CENT.

Percentage is an application of decimals used commonly in business transactions and elsewhere. This process is commonly taught in grades 6, 7, and 8.

(a) **The skills involved in per cent.** The chief skills involved in working with per cents are the following:

1. Changing decimals to hundredths and to per cents
2. Changing per cents to decimals
3. Changing fractions to decimals
4. Changing fractions to per cent
5. Finding a per cent of a number (Case 1)
6. Finding what per cent one number is of another (Case 2)
7. Finding a number with a per cent of it given (Case 3)

In each of the above skills there is included a large variety of applications involving the ability to manipulate decimals of different forms. For example, the changing of decimals to hundredths may be involved in such forms as changing .3, .125, or .12 to hundredths, each form presenting a special application of decimals which must be taught specifically. Finding a per cent of a number involves the multiplication of decimals. Finding what per cent one number is of another involves a knowledge of ratio and the method of reducing a ratio to a decimal. Finding a number when a per cent of it is given involves the division of decimals. Deficiencies in each of these processes may be discovered by means of the diagnostic tests in decimals previously described.

(b) **Types of examples in per cents.** The diagnostic

test[1] on this page makes it possible for the teacher to determine the ability of the pupils to manipulate a large variety of types of examples in percentage. The test is divided into seven parts corresponding to the list of skills above.

BRUECKNER DIAGNOSTIC TEST IN PERCENTAGE

NAME.......................... SCHOOL..................... GRADE............

1. Express the following as per cent:

a. .05 .... *d.* 1.375 .... *g.* 1.2 .... *i.* .125 ....
b. 1.16 .... *e.* .2 .... *h.* $1.33\frac{1}{3}$ .... *j.* 3 ....
c. .25 .... *f.* $.12\frac{1}{2}$ ....

2. Express the following as decimals:

a. 6% .... *d.* 116% .... *g.* 120% .... *i.* $87\frac{1}{2}$% ....
b. 16.5% .... *e.* 40% .... *h.* $166\frac{2}{3}$% .... *j.* 200% ....
c. 15% .... *f.* 118.5% ....

3. Express the following as per cent:

a. $\frac{1}{2}$ .... *d.* $\frac{1}{4}$ .... *g.* $\frac{1}{3}$ .... *j.* $\frac{3}{4}$ .... *m.* $\frac{1}{5}$ ....
b. $\frac{2}{5}$ .... *e.* $\frac{2}{3}$ .... *h.* $\frac{1}{8}$ .... *k.* $\frac{3}{5}$ .... *n.* $\frac{3}{8}$ ....
c. $\frac{1}{6}$ .... *f.* $\frac{4}{5}$ .... *i.* $\frac{1}{10}$ .... *l.* $\frac{1}{20}$ .... *o.* $\frac{7}{8}$ ....

4. Express the following as hundredths and as per cent:

a. $\frac{9}{10}$ = ... = ...% *c.* $\frac{1}{50}$ = ... = ...%
b. $\frac{7}{25}$ = ... = ...% *d.* $1\frac{1}{3}$ = ... = ...%

5. Find the answers to the following:

a. 6% of 80 = *d.* 20% of 65 = *g.* 130% of 800 =
b. 7% of 174 = *e.* 28% of 72 = *h.* 136% of 854 =
c. $12\frac{1}{2}$% of 80 = *f.* $133\frac{1}{3}$% of 60 = *i.* 37.5% of 720 =

6. Find the missing per cents:

a. 8 = ...% of 32
b. 3 = ...% of 10
c. 4 = ...% of 100
d. 15 = ...% of 75
e. 2 = ...% of 40
f. 40 = ...% of 40
g. 30 = ...% of 90
h. 120 = ...% of 96
i. 144 = ...% of 48

[1] Similar diagnostic tests and keyed remedial exercises may be found in *Diagnostic Tests and Practice Exercises*, Grades 6, 7, and 8, published by The John C. Winston Company of Philadelphia.

7. **Find the missing numbers:**

a. 7 = 20% of	*d*. 60 = 100% of ....
b. 12 = 5% of	*e*. 180 = 120% of ....
c. 20 = 20% of	*f*. 255 = 125% of ....

Part 1 is a test of the ability of the pupil to express a variety of types of decimals as per cents. Each example presents a special type of difficulty. The difficulty is obvious in each case. In Table 19 are given the per cents of error on each example made by a group of 405 7A pupils in Minneapolis schools who were given the diagnostic test at the end of the semester. In part 1 of the test, example *a* was worked incorrectly by 8.4 per cent of 405 pupils. These pupils had completed the work in per cent.

An examination of the table will reveal a large variation in the difficulty of the different examples in each part of the test. In part 1 the most difficult types involved the changing of such numbers as 1.2, .125, and 3 to per cents. The introduction of decimals of per cents in part 2 was the cause of many errors in the changing of per cents to decimals; more than half of the pupils made errors in changing 16.5% and 118.5% to decimals. Relatively few errors were made in expressing simple fractions as per cents.

The difficulties present in parts 1 to 3 undoubtedly caused much difficulty in certain types of examples in parts 5, 6, and 7. The per cents of error on each example ranged smaller in exercises involving finding a per cent of a number than in the other two cases. The per cents of error on the whole were greatest in exercises in finding a number when a per cent of it was given, as many as 96.5% of the pupils failing on the example, 255 = 125% of ——. The per cents of error in the other examples

in this part of the test were all very large. It is obvious that practice exercises in percentage must be constructed on the basis of the known difficulty of various types of examples in each of the basic skills involved in per cent.

A preliminary examination of some of the drill materials found in practice exercises and textbooks shows that little consideration has been given to the careful analysis of the types of difficulties that a pupil may encounter in working with per cents. Obviously remedial work, to overcome difficulties revealed by such a diagnostic test as the one on page 239, cannot be adequate unless the teacher has exercises which give practice on the varied types contained in the test and weight the practice according to the difficulty of the processes involved.

TABLE 19

RELATIVE DIFFICULTY OF TYPES OF EXAMPLES IN PER CENT
(Based on Results of 405 7A Pupils)

1. Expressing Decimals as Per Cents:

Example	Per Cent of Error	Example	Per Cent of Error
(a) .05	8.4	(f) 1.33⅓	59.1
(b) .25	8.8	(g) 1.375	61.6
(c) .12½	20.5	(h) .125	63.8
(d) 1.16	29.0	(i) 1.2	64.2
(e) .2	57.5	(j) 3	65.9

2. Expressing Per Cents as Decimals:

Example	Per Cent of Error	Example	Per Cent of Error
(a) 15%	4.2	(f) 116%	25.7
(b) 40%	5.0	(g) 120%	25.7
(c) 6%	12.9	(h) 166⅔%	28.4
(d) 87½%	15.0	(i) 16.5%	63.8
(e) 200%	25.7	(j) 118.5%	65.7

3. Expressing Fractions as Per Cents:

Example	Per Cent of Error	Example	Per Cent of Error
(a) ½	4.2	(d) ⅕	7.4
(b) ¾	5.7	(e) ⅓	7.9
(c) ¼	6.4	(f) ⅖	8.9

TABLE 19 (*Continued*)

(*g*) $\frac{3}{5}$	9.2	(*l*) $\frac{1}{6}$	14.1
(*h*) $\frac{1}{10}$	9.2	(*m*) $\frac{7}{8}$	14.5
(*i*) $\frac{2}{3}$	9.4	(*n*) $\frac{3}{8}$	16.5
(*j*) $\frac{1}{8}$	9.6	(*o*) $\frac{1}{20}$	24.4
(*k*) $\frac{4}{5}$	9.8		

4. Expressing Fractions as Hundredths:

(*a*) $\frac{7}{25}$	30.9	(*c*) $\frac{1}{50}$	48.1
(*b*) $\frac{9}{10}$	36.0	(*d*) $1\frac{1}{3}$	51.1

5. Finding Per Cents of Numbers:

(*a*) $12\frac{1}{2}$% of 80	25.4	(*f*) $133\frac{1}{3}$% of 60	47.8
(*b*) 20% of 65	26.2	(*g*) 130% of 800	49.2
(*c*) 6% of 80	28.2	(*h*) 136% of 854	57.5
(*d*) 7% of 174	30.8	(*i*) 37.5% of 720	73.6
(*e*) 28% of 72	34.1		

6. Finding the Per Cent One Number is of Another:

(*a*) 8 = ...% of 32	61.2	(*f*) 2 = ...% of 40	71.3
(*b*) 30 = ...% of 90	62.5	(*g*) 4 = ...% of 100	72.8
(*c*) 40 = ...% of 40	64.3	(*h*) 144 = ...% of 48	80.7
(*d*) 15 = ...% of 75	65.7	(*i*) 120 = ...% of 96	82.2
(*e*) 3 = ...% of 10	68.9		

7. Finding a Number with a Per Cent Given:

(*a*) 20 = 20% of	78.3	(*d*) 60 = 100% of	89.5
(*b*) 7 = 20% of	78.5	(*e*) 180 = 120% of	95.3
(*c*) 12 = 5% of	84.7	(*f*) 255 = 125% of	96.5

(c) **The most common difficulties in per cent.** In order to determine the most common difficulties in per cent, the test papers[1] of the 405 7-A pupils were studied and the causes of the errors on each of the examples were determined. The errors were classified as (1) those involving percentage as such, and (2) all others, such as errors in computation, miscopying figures, and similar mistakes. The total number of errors analyzed

[1] Amanda Ladenberg and Margaret Harrington, graduate students at the University of Minnesota, assisted in this investigation.

was 11,735. The number of errors in each part of the diagnostic test was as follows:

Part	Percentage Errors	All Others
I	1,770	100
II	1,034	111
III	441	266
IV	798	478
V	340	1,550
VI	2,049	684
VII	1,061	1,053
Total	7,493	4,242

In Table 20 all of the difficulties in each part of the test are analyzed and illustrated, and the number of times each difficulty was found is indicated. The items under each type of difficulty are grouped and have been arranged in the order of their frequency.

TABLE 20

SUMMARY OF DIFFICULTIES IN PER CENT

I. DIFFICULTIES IN CHANGING DECIMALS TO PER CENT

Description	Times Found
1. Drops decimal point and adds % symbol: .2 = 2% or .125 = 125%	650
2. Copies numbers and writes per cent after it: 3 = 3%	528
3. Changes the decimal number to its equivalent fraction and adds per cent symbol: .5 = $\frac{1}{2}$%; .12$\frac{1}{2}$ = $\frac{1}{8}$%	184
4. Omits integers in a mixed decimal and	
(a) Changes only the decimal: 1.33$\frac{1}{3}$ = 33$\frac{1}{3}$%	108
(b) Moves decimal point one place to right: 1.2 = 2%	5
5. Multiplies the whole number by 10: 3 = 30%	46
6. Merely copies numbers: 3 = 3	46

TABLE 20 (*Continued*)

Description	Times Found
7. Moves decimal point to *left:*	
(*a*) One place: $.12\frac{1}{2} = .012\frac{1}{2}\%$	28
(*b*) Two places: $.12\frac{1}{2} = .00125\%$	34
(*c*) Three places: $.12\frac{1}{2} = .000125\%$	3
8. Moves decimal point to *right:*	
(*a*) One place: $.125 = 1.25\%$	41
(*b*) Three places: $.2 = 200\%$	23
9. Changes to equivalent fraction and uses denominator as %: $.05 = 20\%$	14
10. Divides the whole number:	
(*a*) by 10: $3 = .3\%$	11
(*b*) by 100: $3 = .03\%$	13
11. Inserts zero and retains decimal point: $1.2 = 1.02\%$	12
12. Changes part of the decimal to a fraction: $1.375 = 1.37\frac{1}{2}\%$	11
13. Inserts zero and omits decimal point: $1.2 = 102\%$	7
14. Writes a mixed decimal as a common fraction, using the whole number as the numerator and the decimal as the denominator: $1.16 = \frac{1}{16}\%$	5
15. Changes to nearest per cent: $137.5 = 138\%$	1
16. Number of all others including computation errors and omissions	100
Total	1,870
II. Difficulties in Changing Per Cent to Decimals	
1. Merely drops % sign in answer without changing to hundredths: $16.5\% = 16.5$	320
2. Moves decimal point to the left:	
(*a*) one place: $16.5\% = 1.65$	139
(*b*) three places: $116\% = .116$	296

TABLE 20 (*Continued*)

Description	Times Found
(c) three places and contracts to a two-place decimal: $166\frac{2}{3}\% = .16\frac{2}{3}$	10
3. Inserts zeros: $16.5 = 16.05\%$; $16.5 = 16.005\%$	108
4. Drops the % sign and annexes a zero: $16.5\% = 16.50$	68
5. Drops the whole number, inserts a zero, and writes as per cent: $16.5\% = .05\%$	28
6. Changes per cent to fractional equivalents: $87\frac{1}{2}\% = \frac{7}{8}$	11
7. Changes number to nearest hundredths: $87\frac{1}{2}\% = .88$	10
8. Copies number and changes fraction to decimal: $87\frac{1}{2}\% = 87.5$	8
9. Moves decimal point to the right:	
(a) one place: $118.5\% = 1185.$	7
(b) two places: $118.5\% = 11850.$	8
10. Copies number and changes decimal to a fraction: $118.5\% = 118\frac{1}{2}$	7
11. Retains % symbol after changing to a decimal: $6\% = .06\%$	6
12. Drops fraction when per cent contains a mixed number: $87\frac{1}{2}\% = .87$	4
13. Disregards the whole number in the answer: $166\frac{2}{3}\% = .66\frac{2}{3}$	3
14. Changes answer to nearest hundredths: $166\frac{2}{3}\% = 1.67$	3
15. Drops the whole number and changes fractions:	
(a) to its fraction equivalent and used denominator as per cent: $166\frac{2}{3}\% = 3\%$	2
(b) retains decimal: $16.5\% = .5\%$	1
16. Number of all other errors including computation and omissions	111
Total	1,145

TABLE 20 (*Continued*)

III. DIFFICULTIES IN CHANGING COMMON FRACTIONS TO PER CENT

DESCRIPTION	TIMES FOUND
1. Lacks knowledge of fraction equivalents.........	240
2. Adds the symbol % to fraction: $\frac{7}{8} = \frac{7}{8}\%$	73
3. Divides numerator by denominator but carries it out only one decimal place: $\frac{1}{8} = 8\%$	50
4. Multiplies numerator by denominator: $\frac{2}{5} = 10\%$	35
5. Copies the denominator of the given fraction: $\frac{1}{20} = 20\%$	32
6. Copies numerator of fraction: $\frac{7}{8} = 7\%$	5
7. Expresses the correct answer as the denominator of a unit fraction: $\frac{1}{20} = \frac{1}{5}\%$	4
8. Multiplies denominator of fraction by 10 and expresses answer as per cent: $\frac{1}{10} = 100\%$	2
9. Number of all other errors including computation and omissions..............................	266
Total..............................	707

IV. DIFFICULTIES IN CHANGING FRACTIONS TO HUNDREDTHS AND PER CENT

A. Changing fractions to hundredths:

1. Copies numerator and:	
(*a*) moves decimal point two places to left: $\frac{7}{25} = .07$	104
(*b*) moves decimal point one place to left: $\frac{9}{10} = .9$	20
(*c*) moves decimal point three places to left: $\frac{7}{25} = .007$	19

TABLE 20 (*Continued*)

Description	Times Found
2. Copies entire fraction or mixed number and:	
(*a*) moves decimal point two places to left: $\frac{9}{10} = .00\frac{9}{10}$	56
(*b*) moves decimal point one place to left: $1\frac{1}{3} = .1\frac{1}{3}$	6
(*c*) moves decimal point one place to left and changes fraction to decimal: $1\frac{1}{3} = .133\frac{1}{3}$	10
3. Multiplies numerator by denominator and:	
(*a*) points off two places: $\frac{3}{4} = .12$	45
(*b*) points off three places: $\frac{7}{25} = .175$	1
(*c*) places product over 100 as answer: $\frac{7}{25} = \frac{175}{100}$	5
4. Writes numerator as a whole number and the denominator as a decimal: $\frac{9}{10} = 9.10$	34
5. Copies denominator and:	
(*a*) moves decimal point one place to left: $\frac{7}{25} = 2.5$	1
(*b*) moves decimal point two places to left: $\frac{7}{25} = .25$	15
6. Multiplies numerator by 100 and:	
(*a*) retains denominator: $\frac{9}{10} = \frac{9}{10} \times 100 = \frac{900}{10}$	2
(*b*) omits denominator: $\frac{9}{10} = 9 \times 100 = 900$	18
7. Changes mixed number to improper fraction and:	
(*a*) divides denominator by numerator: $1\frac{1}{3} = \frac{4}{3} = 4\overline{)3} = .75$	11
(*b*) copies numerator: $1\frac{1}{3} = \frac{4}{3} = 4$	10
8. Divides numerator by denominator but fails to carry out answer two places: $\frac{7}{25} = 2\frac{20}{25}$ $\frac{9}{10} = .9$	12

TABLE 20 (*Continued*)

Description	Times Found
9. Drops whole number and changes only fraction to a decimal: $1\frac{1}{3} = .33\frac{1}{3}$	10
10. Multiplies both terms of fraction by 10: $\frac{7}{25} = \frac{70}{250}$	9
11. Multiplies numerator by denominator: $\frac{3}{4} = 12$	7
12. Divides denominator by numerator: $\frac{7}{25} = .35\frac{5}{7}$	6
13. Changes denominator to hundredths: $\frac{7}{25} = \frac{7}{100}$	7
14. Multiplies denominator by 10 and:	
(*a*) retains numerator: $\frac{9}{10} = \frac{9}{100}$	4
(*b*) drops numerator: $\frac{7}{25} = 250$	4
15. Places denominator over 100: $\frac{1}{50} = \frac{50}{100}$	4
16. Simply changes mixed number to improper fraction: $1\frac{1}{3} = \frac{4}{3}$	3
17. Simply copies numerator: $\frac{9}{10} = 9$	2
18. Copies denominator and expresses it as a unit fraction: $\frac{9}{10} = \frac{1}{10}$	1
19. Inaccuracy in equivalents: $1\frac{1}{3} = 1.66\frac{2}{3}$	1
20. All other errors including computation and omissions	212
Total—A	639
B. Changing fractions to hundredths and to per cent:	
1. Error due to first step of process being wrong—error in changing fraction to decimal: $\frac{9}{10} = .09 = 9\%$	268
2. Copies original fraction: $\frac{1}{50} = \frac{1}{50}\%$	48

TABLE 20 (*Continued*)

Description	Times Found
3. Copies numerator of first answer: $\frac{7}{40} = \frac{175}{1000} = 175\%$	24
4. Copies denominator of original fraction: $\frac{1}{50} = 50\%$	10
5. Adds numerator and denominator: $\frac{9}{10} = \frac{900}{10} = 910$	9
6. Changes original fraction to decimal: $1\frac{1}{3} = 1.33\frac{1}{3} = 1.33\frac{1}{3}\%$	8
7. First multiplies numerator and denominator of original fraction; expresses result as hundredths and multiplies by original numerator: $\frac{7}{25} = 1.75 = 7 \times 1.75 = 12.25$	3
8. Multiplies hundredths (decimal or fraction) by numerator of original fraction: $\frac{7}{25} = .28 = .28 \times 7 = 1.96$ or $\frac{7}{25} = \frac{28}{100} \times 7 = \frac{196}{100}$	1
9. All other errors including computation and omissions	266
Total—B	637
Grand Total of A and B	1,276

V. DIFFICULTIES IN FINDING A PER CENT OF A NUMBER

Description	Times Found
1. Adds per cent symbol to answer: 20% of 65 = ... Solution: 20% of 65 = 13%	126
2. Divides the numbers:	
(a) Base by the rate: 6% of 80 = ... Solution: $6\overline{)80}$ = $16\frac{2}{3}$	74
(b) The rate by the base: 6% of 80 = ... Solution: $8\overline{)6.00}$ = .75 5 6 40 40	30

TABLE 20 (*Continued*)

Description	Times Found
3. Simply expresses per cent as an equivalent fraction: 37.5% of 720 = ... Solution: 37.5% of 720 = $\frac{3}{8}$	55
4. Uses only fractional part of per cent: $133\frac{1}{3}$% of 60 = ... Solution: $\frac{1}{3} \times 60 = 20$	19
5. Lacks knowledge of equivalents: 37.5% of 720 = ... Solution: 37.5% = $\frac{7}{8}$ $\frac{7}{\cancel{8}} \times \overset{90}{\cancel{720}} = 630$	21
6. Subtracts correct answer from the original number: 6% of 80 = ... Solution: 6% of 80 = 4.80 80 − 4.80 = 75.20	6
7. Drops decimal point in answer: 6% of 80 = ... Solution: 6% of 80 = $.06 \times 80 = 48$	3
8. Merely changes fraction to a decimal: $133\frac{1}{3}$% of 60 = ... Solution: $133\frac{1}{3}$% of 60 = $133.33\frac{1}{3}$	2
9. Multiplies only by decimal fraction: 37.5% of 720 = ... Solution: $.5 \times 720 = 36.00$	2
10. Copies number and points off two places, disregarding per cent entirely: 130% of 800 = ... Solution: 130% of 800 = 8.00	1
11. Divides each number by same divisor and then multiplies: 6% of 80 = ... Solution: 6% ÷ 2 = 3% 80 ÷ 2 = 40 $3 \times 40 = 120$	1
12. All other errors including decimal and computation difficulties and omissions	1,550
Total	1,890

TABLE 20 (*Continued*)

VI. Difficulties in Finding What Per Cent One Number is of Another

Description	Times Found
1. Divides base by percentage: 8 = ...% of 32. Solution: 32 ÷ 8 = 4; 8 = 4% of 32	1,341
2. Fails to express quotient as per cent:	
(*a*) 3 = ...% of 10. Solution: 3 ÷ 10 = .3; 3 = 3% of 10	162
(*b*) 8 = ...% of 32. Solution: 8 ÷ 32 = .25	22
3. Multiplies base and percentage: 8 = ...% of 32. Solution: 8 × 32 = 256; 8 = 256% of 32	112
4. Divides percentage by base but fails to carry work out to hundredths when quotient is a whole number: 144 = ...% of 48. Solution: 144 ÷ 48 = 3; 144 = 3% of 48	75
5. Divides base by percentage and:	
(*a*) multiplies by 10: 8 = ...% of 32. Solution: 32 ÷ 8 = 4; 4 × 10 = 40; 8 = 40% of 32	41
(*b*) multiplies by 100: 8 = ...% of 32. Solution: 32 ÷ 8 = 4; 4 × 100 = 400; 8 = 400% of 32	29
(*c*) and states result as a fraction: 8 = ...% of 32. Solution: 32 ÷ 8 = 4; 8 = ¼% of 32	26
6. Divides base by percentage and then:	
(*a*) divides quotient by 10: 8 = ...% of 32. Solution: 32 ÷ 8 = 4; 4 ÷ 10 = .4	1

TABLE 20 (*Continued*)

Description	Times Found
(*b*) Divides quotient by 100: 8 = ...% of 32 Solution: $32 \div 8 = 4$ $4 \div 100 = .04$..................	40
(*c*) Points off 4 places: 8 = ...% of 32 Solution: $32 \div 8 = .0004$..................	14
7. Expresses percentage as a unit fraction and changes to a fraction equivalent: 2 = ...% of 40 Solution: $2 = \frac{1}{2} = 50\%$..................	39
8. Copies percentage and expresses it as a unit fraction: 8 = ...% of 32 Solution: $8 = \frac{1}{8}\%$ of 32..................	32
9. Attempts to change percentage to a fractional equivalent: 15 = ...% of 75 Solution: $15\% = \frac{15}{100}$..................	31
10. Divides percentage by 10: 4 = ...% of 100 Solution: $4 \div 10 = .4$ $4 = .4\%$ of 100..................	23
11. Changes percentage to a unit fraction and then to its equivalent fraction and subtracts from 100: 4 = ...% of 100 Solution: $4 = \frac{1}{4}$ $\frac{1}{4} = 25\%$ $100\% - 25\% = 75\%$ $4 = 75\%$ of 100..................	17
12. Changes percentage to equivalent per cent and multiplies base: 120 = ...% of 96 Solution: $120\% = 1\frac{1}{5}$ $1\frac{1}{5} \times 96 = 115.2$..................	9
13. Multiplies percentage: (*a*) by 10: 8 = ...% of 32 Solution: $8 \times 10 = 80$ $8 = 80\%$ of 32..................	8

TABLE 20 (*Continued*)

Description	Times Found
(*b*) by 100: 8 = ...% of 32 Solution: 8 × 100 = 800 8 = 800% of 32................	26
14. Subtracts percentage from base and multiplies result by 10: 3 = ...% of 10 Solution: 10 − 3 = 7 7 × 10 = 70 3 = 70% of 10..................	6
15. All other errors including computation errors and omissions............................	684
Total	2,733
II. DIFFICULTIES IN FINDING NUMBERS WHEN THE PER CENT IS GIVEN	
1. Multiplies numbers disregarding decimal form: 7 = 20% of .... Solution: 7 × 20 = 140..................	462
2. Divides the numbers disregarding decimal forms:	
(*a*) Rate by percentage: 7 = 20% of .... Solution: $7\overline{)20}$ = $2\frac{6}{7}$.........................	192
(*b*) Percentage by rate: 20 = 20% of .... Solution: 20 ÷ 20 = 1 20 = 20% of 1................	35
3. Multiplies numbers and points off:	
(*a*) one place: 7 = 20% of .... Solution: 7 × 20 = 14.0.................	39
(*b*) two places: 7 = 20% of .... Solution: 7 × 20 = 1.40.................	154
(*c*) three places: 7 = 20% of .... Solution: 7 × 20 = .140.................	13
4. Copies percentage and:	
(*a*) divides by 100: 7 = 20% of .... Solution: 7 ÷ 100 = .07.................	45
(*b*) multiplies by 10: 7 = 20% of .... 7 × 10 = 70.........................	22
5. Subtracts numbers: 7 = 20% of .... Solution: 20 − 7 = 13...................	37

TABLE 20 (*Continued*)

Description	Times Found
6. Changes per cent to an equivalent and uses denominator as answer: $12 = 5\%$ of .... Solution: $5\% = \frac{1}{20}$; $12 = 5\%$ of 20	14
7. Expresses per cent as: (*a*) an equivalent fraction, but does not complete example: $7 = 20\%$ of .... Solution: $20\% = \frac{1}{5}$	9
(*b*) an equivalent and multiplies by percentage: $7 = 20\%$ of .... Solution: $20\% = \frac{1}{5}$; $\frac{1}{5} \times 7 = 1\frac{2}{5}$	4
8. Divides correct answer by 100: $7 = 20\%$ of .... Solution: $.20\overline{)7.00}$ = 35; $35 \div 100 = .35$	9
9. Doubled per cent, added sum to percentage, and then divided by 10: $12 = 5\%$ of .... Solution: $2 \times 5\% = 10\%$; $10 + 12 = 22$; $22 \div 10 = 2.2$	6
10. Adds numbers: $7 = 20\%$ of .... Solution: $20 + 7 = 27$; $7 = 20\%$ of 27	5
11. Adds per cent symbol to answer: $7 = 20\%$ of .... Solution: $7 = 20\%$ of 35%	4
12. Divides correctly and then multiplies by 100: $7 = 20\%$ of .... Solution: $.20\overline{)7.00}$ = 35; $100 \times 35 = 3500$; $7 = 20\%$ of 3500	4

TABLE 20 (*Continued*)

Description	Times Found
13. Divides rate by percentage and multiplies quotient by 100: $7 = 20\%$ of ______ Solution: $7\overline{)20}$ Quotient $2\frac{6}{7}$ $100 \times 2\frac{6}{7} = 285\frac{5}{7}$	3
14. Adds numbers and divides by 10: $7 = 20\%$ of ______ Solution: $7 + 20 = 27$ $27 \div 10 = 2.7$	2
15. Copies per cent and changes it to mixed number: $180 = 120\%$ of ______ Solution: $180 = 120\%$ of $1\frac{1}{20}$	2
16. All other errors including computations and also omissions	1,053
Total	2,114

The preceding tables contain a detailed analysis of the various kinds of errors that were found in a careful study of the work of seventh-grade pupils. Some of the errors occurred infrequently and are not typical of the kinds of difficulties the teacher can seek to prevent in the teaching of the several processes. Certain errors occur so commonly that the teacher should be thoroughly conscious of them and provide the proper kinds of exercises to reduce such errors to a minimum. Many errors were due to complete lack of comprehension of the processes involved.

The most common errors for each part of the test other than in computation are as follows:

1. In changing decimals to per cents:
   (*a*) Drops the decimal point and annexes the % symbol
   (*b*) Copies number and annexes % symbol
   (*c*) Changes decimal to equivalent fraction and annexes % symbol
   (*d*) Omits integers in mixed decimal and changes decimal to per cent

2. In changing per cents to decimals:
   (*a*) Merely drops % symbol in answer without changing to hundredths
   (*b*) Moves the decimal point to the left incorrectly
   (*c*) Inserts unnecessary zeros
   (*d*) Drops % symbol and annexes zeros
3. Changing common fractions to per cent:
   (*a*) Lacks knowledge of per cent equivalents
   (*b*) Adds % symbol to fraction without changing its form
   (*c*) Divides numerator of fraction by denominator but fails to carry work to hundredths
4. Changing fractions to hundredths and to per cents:
   (*a*) Merely copies numerator and writes as a two-place decimal
   (*b*) Copies entire fraction and writes as hundredths
   (*c*) Multiplies numerator by denominator
   (*d*) Errors in changing fraction to hundredths
5. In finding a per cent of a number:
   (*a*) Adds % symbol to answer
   (*b*) Divides the numbers
   (*c*) Errors in changing per cents to decimals
6. In finding what per cent one number is of another:
   (*a*) Divides base by percentage
   (*b*) Fails to express quotient as per cent
   (*c*) Multiplies base and percentage
   (*d*) Errors due to faulty manipulation of decimals
7. In finding a number with a per cent given:
   (*a*) Multiplies numbers representing rate and percentage
   (*b*) Divides rate by percentage
   (*c*) Divides percentage by rate, disregarding decimal form
   (*d*) Errors in manipulation of division of decimals
   (*e*) Subtracts numbers representing rate and percentage
   (*f*) Inability to manipulate fractions after changing rate to equivalent fraction

An analysis of errors listed above shows that the two chief causes of the difficulties in the processes in percentage are inability to manipulate decimals properly

and a lack of understanding of the process involved. The latter difficulty is clearly evident in the errors listed under finding what per cent one number is of another and finding a number with a per cent given. Many absurd errors would be eliminated if pupils were taught to check the answer to each example. For instance, a solution for the example,

$$3 = \text{..........}\% \text{ of } 10, \text{ such as } 3 = 3\% \text{ of } 10$$

would not be accepted by a pupil who checked his result by finding 3% of 10, thus showing that his answer must be incorrect.

Problems for Study, Reports, and Discussion

1. What are the chief arithmetic skills, other than computation, involved in addition of decimals? in subtraction? in multiplication? in division?
2. List the examples in one of the diagnostic tests in decimals in the order of their difficulty.
3. Analyze the skills involved in the five most difficult examples in each type.
4. Prepare a set of examples that duplicate the types found in one of the diagnostic tests.
5 Analyze a textbook to discover the methods used to teach the pupils to place decimal points in quotients and in products.
6. Give a diagnostic test in one of the processes to pupils in a sixth- or seventh-grade class and analyze the work of the pupils to discover the types most frequently missed and the causes of the difficulties.
7. Select a set of remedial exercises to be used to overcome a specific type of difficulty in decimals.
8. Which do you think are more complicated, processes in common or in decimal fractions?
9. Some authorities believe that the teaching of decimal fractions should precede the teaching of common fractions. What is your opinion on this point?

10. Prepare a set of examples that duplicates those in one of the parts of the diagnostic test in per cents.

11. What are the ten most difficult types of examples in per cent? What skills are involved in each?

12. What are the ten most common faults in per cents?

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CHAPTER VII

DIAGNOSIS IN PROBLEM SOLVING

1. THE MAJOR OBJECTIVES OF PROBLEM WORK IN ARITHMETIC.

The major function of the work in arithmetic is to give the individual the power to apply numbers to the situations in daily life in which the need for numbers arises. Arithmetic was developed by man as a means of giving "order, precision, and arrangement" to those quantitative aspects of environment which otherwise would be chaotic and unorganized. Whatever one reads contains references of a quantitative kind which must be understood before the material being read can be comprehended. Literature abounds with references to various forms and systems of measurement, the meaning of which should be a part of the intellectual equipment of every individual. As a result of the work in arithmetic and in other school subjects, the pupil should acquire a broad background of quantitative information concerning economic, social, industrial, civic, and vocational matters.

Making certain that the pupils acquire an adequate background of quantitative information and that they have practice in quantitative thinking necessary to make this information function effectively, is one of the main objectives of arithmetic. It is obvious that in many situations in life there are occasions when computations of various kinds must be performed. Therefore, the teaching of arithmetic must take into account both the informational as well as the computational applications of number.

In the affairs of life, the need for computation usually arises in connection with some situation or activity. A person determining the amount of change to receive after a purchase has been made, the cost of a number of pounds of a certain kind of food, the cost of gasoline a mile, the score at a ball game, or the profit from the sale of 500 chickens which he has raised, must in each case make one or more computations, depending on the complexity of the situation involved. Usually the data used in the computation must be assembled from various sources, possibly from detailed records of costs, expenses, amounts used, and similar data. Two important elements in arriving at a correct solution of the problem at hand are the care with which the records have been kept, and the ability of the individual to perform the various steps and computations in the process of arriving at a conclusion. Some of the computations can be easily performed while others may be quite complicated and present many opportunities for error.

According to present practice in arithmetic the textbook and certain supplementary drill materials form the basis of the work in the classroom. Textbooks contain the exercises which form the content of the lessons involved in the development of the processes. In addition to these practice exercises, "problems" are given which are intended to give the pupil the opportunity to apply the new processes and those that have been previously presented, in situations approximating those in life. Sometimes these problems are grouped according to the new process being presented. In such cases the pupils often merely manipulate the numbers and pay no attention to the statements in the problem. As one intelligent second-grade pupil said when confronted with such a set of

problems, "These are easy. All that you do is to add the numbers," and then he proceeded to do as he said, disregarding the verbal statements entirely.

Sometimes the problems are grouped under a topic, such as farm problems, problems about chickens, automobiles, making change, and the like. In the newer textbooks, problems relating to a particular situation, such as finding the cost of an automobile trip, the profits from a small garden, or the expenses of the owner of a house, are given. These groups of problems relating to a single situation approximate more nearly life situations than problems grouped by process or by topic. The difference between these groups of problems and actual life situations is that in the problems the complete data are given in an organized way, whereas in real life situations the data would first need to be assembled and organized before a solution could be reached.

2. Application of Processes in Life Situations.

Since the applications of arithmetic as found in real life activities, and as given in most problems in textbooks, present entirely different situations, both types of work should be given in order that the pupil may learn effectively to use his knowledge of how to compute. The teacher must use the opportunities to apply arithmetic in the numerous situations that arise naturally in the every-day activities in the classroom, so that the pupils can see how arithmetic functions in their lives. The work in arithmetic is thereby vitalized and the needed practice on processes motivated and made purposeful. Some of the fruitful sources of occasions for applying arithmetic computations are:

1. Games, counting, keeping score, etc.

2. Measurement and comparison of objects, materials, size, etc.
3. Comparing height and weight with standards.
4. School bank.
5. Paper sales, hobby shows, etc.
6. School lunch, milk supply.
7. Construction, drawing, design.
8. Trips to banks and other places of business.
9. Arithmetic needed in the home, the store, the farm.
10. Problems arising out of geography work, reading, history, and other subjects.
11. Budgets and accounts.
12. How to invest money.
13. How savings banks help us.
14. What it costs to own a home.
15. Illustrations of how we use geometry.

Some of the new textbooks in arithmetic contain many suggested ways in which local applications can be made of the processes or topics that are being presented. While these suggestions are very helpful, they are at best meager and must be supplemented by the skilful teacher as the occasion arises.

Undoubtedly, the best way to teach the pupil how to apply arithmetic is to give him the opportunity to apply arithmetic in real life situations. But under present conditions a large part of the training that pupils are given in that aspect of the subject is by means of the verbal problems found in our textbooks. Practically all of the investigations that have been made of the ability of pupils to apply arithmetic and of the difficulties they have in making such applications have been based on the ability of the pupil to work the types of verbal problems found in textbooks, not upon the ability of the pupil to apply numbers in real life situations.

Problem scales consisting of isolated, verbal problems, often of an inferior quality and arranged in the order of their difficulty, are used to measure the ability of the pupils to solve arithmetic problems. Not a single standard test is available which measures the ability of the pupil to find solutions to situations in which he is not given all of the essential data in an organized way but he must put the needed facts in order himself, as he would in any life situations.

3. STANDARDS FOR EVALUATING PROBLEMS.

The following standards suggest the factors that must be considered in the preparation of problems for the several grades:

STANDARDS FOR THE EVALUATION OF PROBLEMS

1. Problems should arise, as far as possible, from the felt needs of pupils, since "felt needs" are the basis of real problematic situations.

2. The situations on which problems are based should be within the experiences of pupils, especially in the lower grades. In the upper grades some of the important applications must necessarily be on the adult level, although the situations presented must be within the comprehension of the pupils. The processes which the pupils have been taught will naturally affect, and in large measure, determine the contents of the problems he can be given to solve.

3. The vocabulary of the problem should be that to which the pupils are accustomed. The vocabulary of problems can easily be checked by the use of Thorndike's word list.

4. Problems in the lower grades should for the most part include small numbers, since large numbers tend to cause much confusion.

5. Problems for a given grade should be within the comprehension of pupils of that grade. The difficulty of problems will probably have to be determined by experimentation.

6. Problems should be cumulative, that is, the problems

in a group should contain application of both old and new processes and should not be limited to the new process being presented.

7. Problems should do more to the child than merely give practice in computation; where possible they should give the child something to think about, some valuable information, or should illustrate some important application of arithmetic.

8. Problems should be socially significant, that is, according to Thorndike, they:

(a) Should not involve misleading facts and procedures.
(b) Should not involve trivialities and absurdities.
(c) Should not involve useless methods.
(d) Should not involve questions, the answers to which would normally be known.
(e) Should not involve ambiguities and falsities.
(f) Should not involve fantastic situations.

4. Undesirable Types of Problems.[1]

There are so many real problems that can be used in the class work that these undesirable types can easily be eliminated. Teachers should aim to state their problems in arithmetic so that the mental reactions and activities of the pupil will be similar to those used in life itself. It is a questionable practice to give a child fantastic problems to work merely because they will give him practice in some process that can best be given in the regular drill period. Problems should be simple, direct, and real. The following illustrate various types of undesirable problems:

Type I. Problems Involving Misleading Facts and Unusual Procedures:

(a) A box held *144 dozen* oranges. How many oranges were in the box? The usual box of oranges does not hold 144 dozen.

[1] The classification of types of problems is similar to that presented by Thorndike in "Psychology of Arithmetic," pp. 91–95.

(*b*) If you know how much a bushel of potatoes costs, how will you find out how much a peck costs?

The answer to this problem would be found by asking the grocer rather than by making an arithmetical calculation, because potatoes bought in large quantities are usually cheaper per bushel than when bought in small quantities.

Type II. Problems Involving Trivialities and Absurdities.

(*a*) The Smith family took a trip of 203 miles. They were gone a week. What was their average speed per day? The question asked is absurd; the conditions stated are also absurd.

(*b*) If a boy works 16 examples a day, how long will it take him to work 112 examples?

The conditions of this problem are also absurd.

Type III. Problems Involving Useless Methods and Operations.

(*a*) A newsboy earns $\$\frac{2}{5}$ on one day, and $\$\frac{3}{10}$ on another day. How much did he earn in all?

This problem involves a use of fractions which never occurs in real life.

Type IV. Problems Whose Answers Would, in Real Life, Already be Known.

(*a*) Bertha bought a pair of shoes for \$6.00. This was just $\frac{1}{4}$ of what she had in her purse. How much had she before she paid for the shoes?

Before one can state that \$6.00 was $\frac{1}{4}$ of the whole, the whole amount must be known.

(*b*) Jack paid \$4.00 for a pair of skates. This was $\frac{2}{3}$ of all that he had earned. How much had he earned?

Type V. Problems Involving Ambiguities and Falsities.

(*a*) If the price of $\frac{1}{2}$ pint of cream is 25 cents, how do you find the cost of one gallon?

The answer would not be found by using any of the fundamental operations. Cream would be rather expensive per gallon at the same rate we pay per $\frac{1}{2}$ pint.

Type VI. Problems Involving Fantastic Situations.

(a) A dairyman owned 84 cows. He placed an equal number in each of 7 pastures. How many in each pasture?

The answer would not only be known, but it would very rarely, if ever, occur that a dairyman would have 7 pastures and have just the right number of cattle to put an equal number in each pasture, if he ever would wish to do so.

(b) In a certain building there were 400 windows. If there are 4 windows per room, how many rooms are there in the building?

In this problem no allowance is made for halls, or differences in the sizes of rooms.

5. The General Nature of Pupil Difficulties in Problem Solving.

The steps in the process of solving arithmetic problems, according to Monroe,[1] are: (1) reading the statement of the problem with understanding, (2) recalling of principles applicable to the problem, (3) formulating of a plan of procedure concerning the operations to be performed, this being based upon the elements of meaning and the recalled principles, (4) verifying of the procedure which generally does not constitute an explicit step, (5) performing of the operation which is also, strictly speaking, not a step in the reasoning process.

The first step is a complex process involving eye-movement, perception, association of meaning with symbols, and combining the several elements of meaning into an understanding of the problem. From this first step "should come a definition of the problem which is the first step in reflective thinking."

[1] W. S. Monroe, "Derivation of Reasoning Tests in Arithmetic," *School and Society*, Vol. 8, pp. 295–99.

Monroe explains that there are two kinds of words in the statement of a problem. There are words which describe the setting of the problem and words which define quantitative relationships or, as Monroe calls them, "technical words."

In the problem, "What is the value of sugar obtained at a Vermont sugar camp, if it is worth ten cents per pound and six pounds are obtained on the average from each of 1,275 maple trees?" the words "Vermont," "sugar," "maple," and "camp" describe the setting and have nothing to do with the solution of the problem. The technical words such as "value," "per pound," "are obtained," and "each" define quantitative relationships and are cues for formulating the plan of solution. Monroe points out that unless the meaning associated with these words is exact, difficulties will arise.

Monroe illustrates the second step in solving arithmetic problems with the example:

"A man invests $893 in some property. He sells the property for $1,050. What is the rate of profit?"

In this problem it is necessary to recall the principle that the rate of profit is generally calculated upon the amount invested and not upon the selling price.

Monroe's analysis assumes that problems are solved by "reflective thinking." He points out that the difficulty is that pupils do not always reflect. If the problem is familiar, he automatically indentifies it as requiring certain operations. Monroe terms this as a "short-circuiting" of reflective thinking. He also points out another weakness of pupils in arithmetical reasoning when he states, "When the problem is unfamiliar he may try random guessing as the plan of solution."

Stevenson[1] used the interview technique to discover the methods used in problem solving. The following conversation illustrates procedures he discovered:

Problem: When John was 13 years old he received $60 in equal monthly payments for spending money. How much did he receive each month?

Pupil: "I am going to subtract." He put down $60 and said, "No, I have that down the wrong way."

Observer: "Why is it wrong?"

Pupil: "Because I have no number but 13 to put under it."

Observer: "Why can't you subtract 13 from 60?"

Pupil: "It's too little. You can't do it. We never have had that." (This boy never subtracted unless the numbers contained three or more digits.)

Stevenson reports other random, useless procedures used by pupils to solve problems. He describes difficulties of various types and suggests remedial exercises.

Thorndike[2] recognized real need for a diagnosis of pupil difficulties in arithmetic when he wrote:

"In order to block wrong paths most effectively we need to know why the pupil inclines to take them. In order to help him when he is blundering or at a loss what to do, we need to know why he is misled, perplexed, and confused. That is, we need to diagnose his difficulty."

6. Vocabulary as a Factor in Problem Solving.

The necessity of a careful checking of the vocabulary of arithmetic textbooks is made clear by recent investigations in which the Thorndike,[3] Pressey,[4] and other standard lists were used as the basis for the analysis.

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[1] P. R. Stevenson, "Difficulties in Problem Solving," *Journal of Educational Research*, Vol. 11, pp. 95–103.

[2] E. L. Thorndike, *Psychology of Arithmetic* (New York: Macmillan, 1922).

[3] E. L. Thorndike, *Teachers Word Book* (New York: Bureau of Publications, Columbia University).

[4] L. C. Pressey, *The Technical Vocabulary of School Subjects* (Bloomington, Illinois: Public School Publishing Company, 1924).

An excellent example of the results of such a study is reported by Buswell.[1] It is based on the analysis of the vocabularies of six third-grade arithmetic textbooks.

The following quotation gives a summary of the facts discovered:

1. The total vocabulary of six third-grade arithmetic textbooks examined is 2,993 different words.
2. The average vocabulary of the six textbooks is 1,262 different words.
3. The common vocabulary of the six arithmetics is small, there being only 300 words, or 11.7 per cent of the total vocabulary, that occurs in all six textbooks.
4. There are 1,345 words, or 44.9 per cent of the total vocabulary, that are used in only one textbook.
5. The percentage of words which is used in only one textbook ranges in the six textbooks from 11.2 per cent to 21.03 per cent.
6. An average of 32.3 per cent of the words occur only once in the textbooks in which they are used.
7. Of the 1,345 words used in only one textbook, 1,214 are used less than four times in the book in which they occur.
8. Of the total vocabulary of 2,993 words, 328 rank in the first 1,000 for importance in the Teacher's Word Book.
9. There are 2,467 words, or 82.4 per cent of the vocabulary, that rank in the first 5,000 for importance; 284, or 9.5 per cent, that rank in the second 5,000; and 242, or 8.1 per cent, that are not found in the list.
10. The technical vocabulary of the six arithmetic textbooks consists of 306 words, or 10.2 per cent of the total arithmetic vocabulary.
11. Only 34 of the words of the technical vocabulary are used in all of the textbooks, and 75 are used in only one book.
12. Of the 306 words of the technical vocabulary of the arithmetics, 261 rank in the first 5,000 for importance in the Teacher's Word Book.

[1] G. T. Buswell, "Curriculum Problems in Arithmetic," *Second Yearbook of the National Council of Teachers of Mathematics*, 1927), pp. 73–93.

13. There are 514 words, or 17.2 per cent of the arithmetic vocabulary, that are found in the common vocabulary of ten third-grade readers.

14. There are 980 words, or 32.7 per cent of the arithmetic vocabulary, which do not occur in any of the ten readers.

15. There are 1,043 words, or 34.9 per cent of the arithmetic vocabulary, which are used in half or less than half of the readers.

16. There are 616 words, or 20.6 per cent of the arithmetic vocabulary, which occur in only one of the arithmetic textbooks and are not found in any of the readers.

17. Of the 616 words occurring in only one arithmetic and in none of the readers, 292 rank in the first 5,000 for importance in the Teacher's Word Book, 139 rank in the second 5,000, and 185 are not in the list.

18. Of the 616 words mentioned above, 393 occur only once in the textbook, and only 57 occur more than three times.

19. Of the 306 technical words, 164 do not occur in any of the readers, and only 12 occur in all ten of the readers (Chapter V).

Buswell comments on these data as follows:

The facts presented in the preceding summary indicate that an arithmetic textbook would present considerable difficulty to children if the only requirement were that it be read and if the sole criterion of difficulty were the number of new words encountered.

A third of the words of the six textbooks occur only once, no provision being made for sufficient repetition to make them familiar to the child.

A third of the words in the arithmetics do not occur in any of the ten readers. The teacher of arithmetic, therefore, becomes responsible for teaching these new words in the arithmetic class. Even in respect to the technical vocabulary used by textbooks in arithmetic, there has been little common practice, only 34 of the 306 technical words being common to all of the six books.

The situation disclosed in the preceding summary is still more striking when one observes the variation in practice for single sets of textbooks. Authors have simply drawn upon their own private vocabulary without regard for the difficulties which the limitations of their vocabulary may cause children. One of the first problems which must be solved before the reading difficulties encountered in arithmetic are removed is a standardization of the vocabularies used in the different books and a correlation, grade by grade, between the vocabularies used in arithmetics and the vocabularies encountered in general reading in the elementary school.

7. Causes of Difficulties in Problem Solving.

Stone[1] states that "all arithmetical ideas, to exist, must have a background of mental images which are outgrowths of experience. Hence, the school must provide broad, concrete, meaningful experiences, either in or out of school, in which quantitative thinking is a major element. Failure to provide these clear mental images, which the statement of a problem should call up in the child's mind, is perhaps the chief source of failure to solve a problem."

In other words, the problem is not concrete to the child unless he is able to form a clear mental picture of the situation described. It may be that he has not read the problem carefully or that he may have read it carefully and yet, through lack of experience in the situation described, he may not be able to form a picture of the situation.

Stone reports a problem in which a girl found "the cost of two pencils if one cost five cents," but could not find the cost of two meters of silk if one meter cost five francs.

[1] J. C. Stone, *The Teaching of Arithmetic* (Chicago: Sanborn, 1922), pp. 170–89.

Upon being questioned the child said, "I don't know what a meter is and I don't know what francs are, so of course, I can't solve it."

Stone believes that other principal sources of failure in problem solving are:

1. Problems are not concrete.
2. There is inaccuracy in computation.
3. Approximations were made before the computation was completed.
4. There is lack of understanding of the fundamental processes.
5. There is an assumption, by the author, of facts not known to the pupils.
6. Difficulty of problem, that is, the relations existing between data are too complex and as a result the pupils lack the mental power to reason out what process to use.

"The failure on the part of pupils to solve arithmetic problems is due to poor reading and consequent inability to understand the problems" was reported by Estaline Wilson[1] in 1922.

Teachers have not called upon pupils to master the meaning of arithmetic problems, and as a result, pupils feel no responsibility for remembering any of the "reading facts of the problems." This has come about because of the fact that most problems are stripped of details which make reading vivid. Miss Wilson points out that the color, style, and material of a dress are quite as important to the purchaser as the price but the authors of arithmetic textbooks do not take such associated ideas into consideration in the preparation of problems.

Another view of the problem is given by Hunkins and

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[1] Estaline Wilson, "Improving the Ability to Solve Problems," *Elementary School Journal*, Vol. 22, pp. 380–86.

Breed.[1] They feel that the ability to solve problems depends upon something more than reasoning ability if the vocabulary employed is difficult or if the computational processes are technical. They suggest that if vocabulary, understanding of problems (comprehension of sentences and paragraphs) and ability to use technical processes (computational skill) were measured by separate tests, the exact difficulty of a pupil could be found.

The most comprehensive study of this problem was made by Superintendent Banting[2] in the Waukesha Public Schools in 1922-23. The study which was carried on in the grades below the Junior High School was begun by giving Monroe's and Buckingham's Reasoning Tests. The results of these tests were carefully weighed, the daily work of the pupils was observed, and each pupil was questioned as to his difficulties. In fact, Banting's report regarding the difficulties in arithmetical reasoning seems wholly inclusive, but, as he states, further investigations will extend and substantiate the evidence of causes of failure. Banting's technique is quite similar to that employed by Buswell and John. This study seems so vitally important that a brief summary of the causes of failure will be reported. They are as follows:

I. Failure to comprehend the problem in whole or in part which might be due to lack of ability in silent reading, inability to read the language of arithmetic, carelessness in reading, or lack of the necessary experience to reproduce mentally the concrete situation of the problem.

II. Lack of ability to perform accurately and readily the fundamental operations.

[1] R. V. Hunkins and F. S. Breed, "The Validity of Arithmetic Reasoning Tests," *Elementary School Journal*, Vol. 23, pp. 453-66.

[2] G. O. Banting, "The Elimination of Difficulties in Reasoning," *Second Yearbook of the Department of Elementary School Principals*, (Washington, D. C., 1923), pp. 411-21.

III. Lack of knowledge of facts essential to the solution of a problem.

IV. Lack of ability to identify the proper process or processes with the situations indicated in the problem.

V. Lack of sufficient interest in the problem to inspire the required mental effort.

VI. Failure to form the habit of verifying the results.

VII. The habit of focusing the attention upon numbers and being guided by them instead of by the condition of the problem.

VIII. Pupils are sometimes completely nonplussed by large numbers.

IX. The habit of being guided by some verbal sign instead of by making an analysis of the problem.

X. Lack of ability or care to properly arrange the written work in orderly, logical form.

XI. The failure to recognize the mathematical similarity to type problems which the pupils understand, because of some unusual situation in the problem in question. For example, the pupil who readily solved problems dealing with the purchase and sale of familiar things, failed when given a problem dealing with the purchase and sale of a farm.

XII. Lack of ability to understand quantitative relations such as: (*a*) cost, loss or gain, selling price, (*b*) income, expenditures, amount saved, etc.

XIII. The pupil may fail because the problem requires exertion beyond his span of attention.

XIV. The pupil may fail because of absolute inability to do reflective thinking.

Osburn[1] reports a study made in 1922 to discover the weak spots in arithmetical reasoning. The Buckingham Problem Test was given to 6,000 children in eighteen counties and one large city and the 30,000 errors made were classified as follows:

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[1] W. F. Osburn, *Corrective Arithmetic*, Vol. 1 (Boston: Houghton Mifflin Company, 1924), p. 38.

		Per Cent
1. Total failure to comprehend the problem...	30	
2. Procedure partly correct but with the omission of one or two essential elements....	20	
3. Failure to respond to fundamental quantitative relations......................	10	60
4. Errors in fundamental processes...........	20	
5. Miscellaneous errors.......................	2	
6. Errors whose causes could not be discovered	18	40
Grand Total..............................		100

Osburn's discussion regarding the weaknesses children have in arithmetic is very absorbing. He states, "Industrial concerns find it profitable to study the situations in which their output falls short of the expected amount. Few salesmen talk about the weak points of what they sell, but it is safe to assume that all good salesmen have a pretty definite knowledge of what these weak points are. Teachers, however, are almost always destitute of such information."

That Osburn believes much scientific information of genuine value can be gleaned from individual investigations of pupil difficulties can be inferred from the following: "Now, a study of the errors made by a thousand or more children can be carried on only with the coöperation of many people. On the other hand, a study of the errors of fifty or a hundred children can easily be made by one person. The fact that such a study reveals typical errors gives much significance to the work of individual investigators who are dealing with small groups. Progress, therefore, should be easy in the future."

Osburn states that there are two types of educational disabilities—one that must be outgrown if it is ever removed and the other that may be removed very easily

if its nature and causes can be discovered. The former is due to the lack of mental ability on the part of the child for which the teacher is not to blame. The latter is due to poor teaching, poor health, etc., part of the responsibility for which can be laid at the door of the teacher and her predecessors.

According to Osburn, the most frequent trouble that a child has in problem-solving is his inability to identify the proper process to use. "He looks at his teacher and asks artlessly, 'Do you subtract?'

"On examination or test day the teacher, of course, does not answer. . . . There are four fundamental processes, and the child has one chance in four in one-step problems of getting the correct answer by mere guess. He will guess wrong, however, about three times out of four.

"Thus, in the problem, 'We learn to spell two new words a day in our school. How many new words do we learn in eight days?' We find by far the largest number of wrong answers to be ten, six, and four. These represent wrong answers. The children add, subtract, or divide because they are unlucky. It follows, of course, that some of the children who get the right answer have guessed luckily. In problems of two or three steps the same thing occurs, but it is not so easily detected."

Osburn believes that another large source of error arises from the misreading or misunderstanding of a portion of the problem by the child. In the problem, "A coat cost ten times as much as a hat. Both together cost $66.00. How much did the coat cost?" the pupil obtained an answer of $6.60. He left the coat entirely out of consideration and attempted to find the cost of the hat. This is an excellent example of the above mentioned source of error.

A third type of error arises from an ignorance of fundamental relations such as those of income and outgo, profit and loss, etc. Osburn's advice regarding the location of types of errors in a particular problem is pointed and specific.

"It is recommended that the teacher attempt first to account for failure on the part of the child by the inspection of his wrong answers. If doubt still remains, be sure to have him do the work aloud. Having thus gotten a correct notion of the true nature of the child's difficulty, it is usually easy to remove it."

8. Specific Diagnostic Procedures in Problem Solving.

The diagnostic procedures which have been discussed for locating difficulties in computation can also be applied to problem solving. These procedures are:

1. Analysis of the pupil's written work.

2. Observation of the pupil's work when the steps are given aloud.

3. Exercises for the location of specific types of deficiencies. The third procedure is as yet undeveloped.

(a) **Analysis of the pupil's written work.** In many cases it is possible to determine from an analysis of the pupil's written work the cause of difficulty in a particular problem. Mistakes in computation or failure to complete the whole problem are easily detected. Osburn suggests that it is easy to determine whether the pupil comprehends the problem by an analysis of the answer. If in working the problem, "I have 8 cents. How many apples can I buy at 2 cents each?" the pupil gives 10, 16, or 6 as the answer. In each case the wrong process was used to get the answer to the problem, clearly indicating

inability to select the correct process. Answers such as 2 or 3 indicate probable errors in computation. Two-step problems are often not completed, resulting in an incorrect answer; for example, the answer given in the two-step problem, "I buy 2 apples at 4 cents apiece. What change should I receive from a quarter?" may be given as 8 cents, which is found by multiplying 2 × 4, and failure to subtract the product from 25 cents.

A typical analysis[1] of the work of a 6A class of 37 pupils in the Buckingham Problem Scale is contained in Table 21.

TABLE 21

DISTRIBUTION OF DIFFICULTIES FOR EACH PROBLEM ON THE BUCKINGHAM PROBLEM SCALE BY A MINNEAPOLIS 6A CLASS GRADE

Problem Number	Entirely Correct	Correct Method; Error in Fundamentals	Wrong Method; Correct Fundamentals	Method Incomplete; Correct Fundamentals	Method Incomplete; Error in Fundamentals	Method Wrong; Error in Fundamentals	Number of Pupils Attempting
1	32	..	3	..	..	2	37
2	35	1	1	..	..	..	37
3	34	..	..	..	..	3	37
4	27	4	6	..	..	..	37
5	15	..	17	..	..	5	37
6	29	7	1	..	..	..	37
7	29	..	3	..	..	5	37
8	9	..	..	28	..	..	37
9	32	2	..	..	..	1	35
10	16	12	2	..	..	7	37
11	20	8	2	1	..	6	37
12	29	..	2	..	..	6	37
13	22	1	4	..	..	10	37
14	10	7	4	..	..	13	34
15	13	1	9	..	..	12	35
Total	352	43	54	29	0	70	548
Per Cent of Attempts	64	8	10	5	..	13	100
Per Cent of Errors	..	22	28	14	..	36	..

[1] Reported in J. C. Brown and L. D. Coffman, *The Teaching of Arithmetic* (Chicago: Row Peterson, 1924), p. 53.

There are 15 problems in the test. Problem 1 was worked correctly by 32 of the 37 pupils; 3 pupils used the incorrect method of solution but made no errors in their computations; 1 pupil used an incorrect method and also made errors in the solution. There were 43 cases, 8 per cent of the attempts, in which the pupil used the correct method in solving one of the problems but made an error in computation; 54 cases, 10 per cent of the total, in which the wrong method was used but the computations were correct; in 29 cases, or 5 per cent of the total, the method was incomplete, but the computation was correct as far as the work had proceeded; in 70 cases, 13 per cent of the total, the method used was wrong and there were also errors in computation.

An analysis such as this points out the fact that errors of certain kinds were made but gives no evidence as to *why* they were made in most cases. In 58 per cent of all problems worked incorrectly there was an error in computation; in 14 per cent of the cases, the method was incomplete; in 64 per cent of the cases the wrong method was used. The reasons for the errors must be determined when not obvious by an individual examination.

(b) **An observation of the pupil's work when the steps in the solution are given aloud and the difficulties are revealed by an interview.** Almost all of the studies of pupil difficulties in problem solving have been based on an analysis of the written work of the pupil. The deficiency of this method was pointed out on page 277.

The studies of Banting and Stevenson, previously discussed, were based directly on the results of a careful observation of the pupil's work, and interviews to determine special difficulties not otherwise diagnosed. Neither of these studies gave any information as to the relative

frequency of each kind of error or difficulty, although their procedures have been used as the basis of subsequent studies, one of the most significant of which is that of Martin.[1]

Martin made an analysis of the problems in the Buckingham Scale for Problems in Arithmetic and Stanford Reasoning Tests worked incorrectly by a group of 56 pupils markedly deficient in problem solving. This analysis revealed their common types of difficulties. These difficulties were classified as they were discovered. The diagnostic process used by Martin consisted of making an analysis of the pupil's work in a problem to discover the cause of error, if possible, and verifying the findings by having the pupil rework the problem and asking him questions concerning it. In all, 1,014 mistakes in these two tests were analyzed in this manner. The difficulties which the pupils encountered are listed in Table 22.

TABLE 22

A TABULATED SUMMARY OF THE DIFFICULTIES IN THE BUCKINGHAM AND STANFORD REASONING TESTS FOR THE FIFTY-SIX PUPILS IN MARTIN'S STUDY

Cause of Difficulty	Errors							
	4th Grade		5th Grade		6th Grade		Totals	
	Number	Per Cent	Number	Per Cent	Number	Per Cent	Number	Per Cent
1. Failure to comprehend in whole or in part	83	30.3	62	30.1	107	20.0	252	24.8
2. Carelessness in reading	26	9.5	35	17.0	109	20.4	170	16.8

[1] C. R. Martin, "An Analysis of the Difficulties in Arithmetical Reasoning of Fourth, Fifth, and Sixth Grade Pupils," *Master's Thesis*, Unpublished (Minneapolis, Minnesota: University of Minnesota, 1927).

TABLE 22 (*Continued*)

Cause of Difficulty	Errors							
	4th Grade		5th Grade		6th Grade		Totals	
	Number	Per Cent	Number	Per Cent	Number	Per Cent	Number	Per Cent
3. Inability in the use of fundamentals........	36	13.2	31	15.1	98	18.3	165	16.3
4. Confusion of processes	67	24.4	25	12.1	63	11.8	155	15.3
5. Lack of knowledge of facts essential to the solution of the problem..............	11	4.0	5	2.4	31	5.8	47	4.6
6. Inability in the use of decimals............	0	0.0	6	2.9	39	7.3	45	4.4
7. Carelessness in arranging problems....	2	0.7	14	6.8	15	2.8	31	3.1
8. Lack of sufficient interest..............	3	1.1	4	1.9	24	4.5	31	3.1
9. Inability in the use of fractions...........	9	3.7	9	4.4	12	2.2	30	2.9
10. Ignorance of quantitative relations......	0	0.0	7	3.4	18	3.4	25	2.5
11. Could not analyze...	36	13.1	8	3.9	19	3.5	63	6.2
Totals..........	273	100.0	206	100.0	535	100.0	1,014	100.0

Of the total of 1,014 errors which were found in these two tests, 273 were made by the fourth-grade pupils, 206 by the fifth-grade pupils, and the remainder, or 535, by sixth-grade pupils. The percentages indicate that the fourth- and fifth-grade pupils did not comprehend as readily as the pupils in the sixth grade. The pupils of the two upper grades read less carefully than those in the fourth grade.

Inaccuracy in fundamentals increased the totals in the sixth grade, making the largest percentage of errors. The fourth-grade pupils confused their processes in solving far more than the other pupils, 24.4 per cent of their errors being made in this way. An illustration of this

confusion is a selection of multiplication, addition, or subtraction to solve a problem when the correct process is division.

The sixth-grade pupils showed lack of interest in problems a far greater proportion of times than the other pupils.

Difficulties in thirty-six of the fourth-grade problems could not be analyzed. Twenty-five of these were incorrectly worked by two pupils, seventeen by one, and eight by the other.

Perhaps the fourth-grade percentage would not have been so big if it had not been for this rather singular circumstance. There was not a large enough frequency in the other causes of error to make conclusions sufficiently valid to be of definite value.

9. Description and Symptoms of Faults.

A discussion of the several causes of difficulty as described by Martin and the types of responses that indicate each follows:

(a) *Failure to comprehend in whole or in part.* The most frequent cause of difficulty was failure on the part of the pupil to comprehend the problem in whole or in part. This might be due to a general inability in silent reading or to insufficient experience on the part of the pupil, either actual or otherwise, to be able to reproduce the situation of the problem in his own mind. Also, the pupil might comprehend the greater part of the problem but fail because he did not understand or have in his working vocabulary some of the arithmetical terms used in the problem proper.

In the problem, "If you can get 3 ginger-bread dogs for 5 cents, how many can you get for 10 cents?" one lad

divided the 10 by 5 and stated that 2 could be secured for 10 cents. When asked why he did this he stated that the ginger-bread dogs were 5 cents apiece and therefore $10 \div 5$ equaled 2. The writer thought, perhaps, he had read the problem carelessly a second time and called to his attention the words, "3 ginger-bread dogs for 5 cents." The boy stated that couldn't be and that he hadn't had any problems like that. He either did not comprehend or could not visualize the problem because the situation was new to him.

In the problem, "Henry gathered 5 quarts of nuts. He sold then at 8 cents a quart and spent the money for oranges at 4 cents apiece. How many did he buy?" one boy added the 5, 8, and 4 and gave 17 as the result. When asked why he did that he said he did not know what else to do, indicating that he did not comprehend what was wanted nor how to go about securing it.

One of the problems of the Stanford Reasoning Test reads as follows: "If a train goes 60 miles in 3 hours, how far does it go in one hour?" One girl added the 60 and the 1 to get 61 for an answer. She could not explain why she did it that way and further questioning revealed that she did not understand the problem; she added two convenient numbers and let it go at that.

Many pupils use this hit-or-miss method of securing an answer when they do not know what to do. They seem to think that an answer of some kind, no matter how incorrect it is, will help to shield them from further consequences.

In the problem, "Henry was marked 87 in geography the first month, 91 the second, and 93 the third month. What was his average grade?" one girl added the 87, 91 and 93 but did not know how to proceed from that

point. She did not understand the significance of the words "average grade." In other words, she did not understand one of the technical words of arithmetic. Lack of an adequate arithmetic vocabulary and vague quantitative concept are two of the chief causes of lack of comprehension.

(*b*) *Carelessness in reading.* The next most frequent cause of difficulty was caused by carelessness in reading. A child may be able to read and solve a problem and yet through careless reading solve the problem incorrectly. One pupil in working the problem, "A boy owned three kites, each of them having 150 feet of string. How many feet of string had he?" read the 150 as 1.5 and multiplied it by 3. Had she read the problem carefully, she would have been able to solve it correctly.

Another problem reads, "Find the total of two one-dollar bills, three five-cent pieces, two dimes, and three quarters." A girl added these units correctly with the exception of the dollar-bill item. In reading, she had skipped over the "two" or omitted it in some way and added one dollar rather than two dollars to secure the sum. Her result was, of course, too small by one dollar.

One other illustration of this all too prevalent difficulty. In the problem, "A man pays the street-car fare for himself and two friends. If the fare is 7 cents, how much should he receive from a half dollar?" one child subtracted the 7 from 50, to obtain a result of 43 cents. When her attention was called to the words "and two friends" she said she had not noticed these words.

Under this heading might also be classified the failure to recognize the arithmetical similarity to other problems because of some new element in the problem. For example, one boy was able to work the problem, "How

many eggs are there in 7 nests if each nest has 3 eggs?" but said he did not understand a problem of similar type reading, "How many cents will 8 oranges cost at 3 cents each?"

(c) *Inability in the use of fundamentals.* Inability to use the fundamental processes correctly proved the nemesis of many. One hundred sixty-five errors were made in either addition, subtraction, multiplication, or division. The tables in the preceding chapter indicated that most pupils needed drill in one or more of the four fundamental processes, and it is only natural that this lack of ability would hamper these pupils in the solution of problems where the combinations they do not know are present.

In the problem of the kites which has been quoted before, one child multiplied 150 by 3 and obtained 490, a clear case of inability in multiplication.

One of the problems in the Buckingham Test reads, "If an electric car runs 9 miles an hour, how many hours will it take to travel from one city to another 117 miles away?"

One girl selected the correct process, division, but when she divided 11 by 9 she obtained a remainder of 3 rather than 2. Her problem was worked in this fashion:

$$\begin{array}{r} 1\ 4\tfrac{1}{9} \\ 9\overline{)11^{3}7} \end{array}$$

her result was $14\frac{1}{9}$. This incorrect result was caused, perhaps, by carelessness in division rather than inability.

Another one of Buckingham's problems reads, "23 children belong to our class, but only 19 are present. How many children are absent?"

One 4B girl subtracted 19 from 23 and obtained *3*, a clear case of correct process but inability in subtraction.

The three preceding problems have illustrated mistakes in multiplication, division, and subtraction.

The following problem will illustrate difficulty with addition. This problem, one of those in the Stanford Reasoning Test, reads, "David earned $3.50 in June, $2.25 in July, and $1.50 in August. How much did he earn in all?"

A boy in grade 5B added the three items to obtain $8.25 rather than $7.25, a clear case of correct process but incorrect addition.

These difficulties in the fundamental processes can be largely eliminated by thorough remedial instruction and drill.

(*d*) *Confusion of processes.* Confusion of processes caused many pupils to arrive at incorrect results. That is, while a pupil may understand the different processes, he may not know which process to use for a given problem, whether to add or subtract, or multiply or divide.

One boy in grade 4B worked ten problems incorrectly because he did not know what process to use in the solution of his problems. He seemed to guess at the process to be used. If he guessed correctly, all was well and good; otherwise, the problem was wrong. It might be well to state a few of the problems this boy worked in this fashion.

The third problem of the Stanford Reasoning Test reads like this, "A hen had 9 chicks and 3 of them died. How many were left?"

The boy, instead of subtracting, added to get a result of 12.

Another, "How many eggs are there in 7 nests if each nest has 3 eggs?"

The proper process necessary to the solution of this problem is, of course, multiplication. This boy added

and secured ten. A correct use of the process but incorrect selection of process. He worked the eighth problem involving the finding of the cost of 8 oranges at 3 cents each in exactly the same way; he added instead of multiplying. In the eleventh problem, in which he was to find the cost to each of five girls of a present purchased by all for 25 cents, he multiplied instead of dividing, securing $1.25 rather than the correct result, 5 cents.

In the Buckingham Reasoning Test the same boy did precisely the same thing in four problems, adding in the first problem where he should have multiplied, adding in the second where he should have subtracted, adding in the fifth instead of multiplying, and multiplying in the sixth instead of dividing. This boy was normal in the Stanford Fundamentals Test and slightly below normal in the Woody-McCall Scale. Apparently, he had never learned to adapt himself to a given problem or situation or to select the proper process for solution.

(*e*) *Lack of knowledge of facts.* Several pupils were unable to work problems where it was necessary to be familiar with certain tables, arithmetical facts, etc., to be able to arrive at a solution.

For example, in the problem, "How many dimes are there in a dollar?" One pupil stated that she did not divide, but actually thought there were 9 dimes in a dollar.

In another problem reading, "If a fence rail is 10 feet long, how many rails will it take to reach a mile?" her result was 120. She did not know the number of feet in a mile.

In another problem of the same test reading, "If the butcher's scales read one ounce too much on each weighing, how much is a customer overcharged on a pound of

steak at 48 cents a pound?" several sixth-grade students could not find the correct solution because they did not know the number of ounces in a pound.

A 5B-grade pupil in Buckingham's problem, "An automobile was run 30 miles every day for a week. How many miles did it go?" did not know the number of days in a week. He multiplied by 5 rather than by 7, the number of days in a week.

More illustrations of this nature could be given had the tests contained more problems where a knowledge of tables was required for solution. The remedy here evidently lies in more thorough drilling on the essentials and important relationships. Frequently, a textbook contains a table of measures with several problems following it and bearing on the contents of the table. From then on the facts in the table may not be reviewed by the author throughout the remainder of the book, or at best mentioned only rarely. This type of textbook should be purposely avoided if we are to eliminate the errors caused by this neglect.

(*f*) *Inability in the use of decimals.* Inability in handling decimals correctly caused 45 errors, or 4.4 per cent of the total. The tenth problem in the Buckingham Test involves the addition of several addends, the items being expressed in dollars and cents. Many of the sixth grade pupils, and of course, some in the lower grades who could not be expected to be proficient in the use of decimals, confused the dollars and cents columns in the addition and naturally secured a variety of results. For the same reason a large number of students made the same type of error in the sixth problem of the Buckingham Test which has already been stated. This is also a problem in the addition of a column involving cents and dol-

lars and many had not learned how to keep the decimal points, the cents and the dollars, lined up with each other.

In the problem, "If 0.78 of the weight of potatoes is water, how many pounds of water are there in a bushel of potatoes, if a bushel of potatoes weighs 60 pounds?" many of the pupils multiplied properly but either pointed off incorrectly or did not point off at all. This type of difficulty can be eliminated in most cases by thorough, efficient instruction in the use of the decimal point.

(g) *Inability in the use of fractions.* Lack of knowledge of how to handle fractions correctly proved the downfall of many pupils. In the problem, "James has 28 marbles. He gives half of them to Charles. How many has he left?" many pupils gave 16 as the answer, indicating that they did not know how to take one-half of 28.

Several pupils missed one of the Buckingham problems in which the pupil was given the weight of two tubs of maple sugar as 42 pounds and asked to find the weight of one of the tubs if the other weighed $18\frac{1}{4}$ pounds. The error occurred when the pupil attempted to subtract $18\frac{1}{4}$ from 42. One pupil did not know what to do with the $\frac{1}{4}$, so he dropped it and subtracted 18 from 42.

Buckingham's twelfth problem reads, "How many weeks will it take Joseph to save 21 dollars for a bicycle if he saves $1\frac{1}{2}$ dollars a week?"

One boy explained that he did not know how to divide 21 by $1\frac{1}{2}$ so he divided 21 by 1. Ignorance is usually responsible for freakish solutions.

In the problem, "A boy had 210 marbles. He lost one-third of them. How many were left?" Some pupils got rather inexplicable results when they multiplied 210 by one-third. One pupil got 51, another $133\frac{1}{3}$, and

when asked how they secured these results, these pupils did not seem able to explain how they were obtained. They simply did not understand the use of fractions.

Results of this kind do not seem to be the fault of the children; they are rather, I should say, in many instances the results of inefficient instruction, and poorly written textbooks.

(*h*) *Carelessness in arranging problems.* A lack of neatness and orderly arrangement of figures caused some pupils to fail. A few insisted upon splashing figures all over the page and while, in some cases, they seemed to know what process to use and how to use it, they were confused, apparently, by the disorderly arrangement before them and often used the wrong figures after the correct result had been secured.

(*i*) *Ignorance of quantitative relations.* Ignorance or lack of ability in quantitative relationships such as cost, gain, selling price, etc., caused some errors. Only two of the problems given, the tenth and eleventh in the Buckingham Test, involved relationships such as those described above.

The tenth problem reads, "A store takes in the following sums: $1250, $300, $175, $16.25, $120.50, $32.75, $68.50. It pays out $600, $360, $166.67, $33.33, $240. How much remains after payments are made?"

A few of those who attempted this problem did not know what to do after adding the two columns. Some added the entire set of figures as one column; others found the two sums and added them together instead of subtracting the one from the other. They evidently did not understand the relationship between selling price, cost, and profit, of which this is a type problem.

The eleventh problem is as follows: "A man bought a house for $7,250. After spending $321.50 for repairs, he sold it for $9,125. How much did he gain?"

A few pupils added the $7,250 and $321.50 as they should have done and then tried to subtract $9,125 from this sum. One pupil added $9,125, the selling price, and the cost. Still another subtracted the cost from the selling price without considering the expense for repairs. Another subtracted the expense incurred from the cost and then attempted to subtract the selling price from this result. These illustrations show that the pupils did not understand the quantitative relationships involved.

The pupils who made the mistakes mentioned were 6A pupils and should have received instruction along these lines. They may have received it and not absorbed it.

(*j*) *Lack of interest.* There were thirty-one cases of lack of interest which were detected. Dawdling, lack of effort, and inattention were the chief symptoms. This proportion is a very large one for the small group concerned. Undoubtedly in a large, unselected group the lack of interest would not be such a prevalent characteristic.

(*k*) *Errors which could not be analyzed.* Several of the mistakes could not be analyzed even with the help of the pupils. They did not seem to know how or why they secured the results they did and did not work the problems the same the second time.

One 4B pupil secured 2 as a result for the problem, "How many are 3 eggs and 2 eggs?"

Another problem, "Mary is 7 years old. How old will she be in 3 years?" the same girl secured 8 for a result.

To the problem, "If you can buy a pencil for 4 cents

and pay for it with a dime, how much change should you get?" her written response was "21 each." She could not explain it afterwards. She stated that there were 18 dimes in a dollar, and in answer to the eleventh Standard Reasoning Test Problem, "Five girls buy a present costing 25 cents. How many cents does each pay?" she gave 10 as a result.

In the Stanford Fundamentals Test she seemed to be able to add in most cases, but when asked to subtract she usually added or secured the incorrect answer. When she subtracted 25 from 96 she secured 61 rather than 71; when she multiplied 2 by 3 she secured 5 as a result. She divided 6 by 2 correctly but when asked to divide 8 by 4 her result was 3. Her scores in the addition, subtraction, multiplication, and division sections of the Woody-McCall Scale were 18, 13, 8, and 1, respectively, as compared with medians of 17.8, 15.7, 11.0, and 10.5.

She seemed to be able to do any of the processes excepting division, if she could work a number of successive problems in one process without switching to another, but if processes were switched rapidly her results were usually erratic.

This girl had an I.Q. of 88. She was 4 months above standard in the Stanford Reading Test, and 46 and 14 months, respectively, above standard in the rate and comprehension scores of the Monroe Silent Reading Test. The evidence would point to insufficient training in the fundamentals and inability to adapt herself to swift changes from one process to another.

In the other cases where the results could not be analyzed, with the exception of one which will not be recorded until later, it appeared that the pupils felt they had to give some response and guessed at the results.

There was utterly no connection between the correct results and those obtained, and later quizzing of the pupils revealed that they did not understand the problems and did not get the same results a second time.

10. Miscellaneous Factors Causing Difficulties in Problem Solving.

As the result of a recent study of methods used by pupils in solving problems, Monroe[1] comes to the following conclusion:

> A large per cent of seventh-grade pupils do not reason in attempting to solve arithmetic problems. Relatively few pupils follow the plan described on page 7 (in the report). Instead, many of them appear to perform almost random calculations upon the numbers given. When they do solve a problem correctly, the response seems to be determined largely by habit. If the problem is stated in the terminology with which they are familiar and if there are no irrelevant data, their response is likely to be correct. On the other hand, if the problem is expressed in unfamiliar terminology, or if it is a "new" one,—relatively few pupils appear to attempt to reason. They either do not attempt to solve it or else give an incorrect solution.

Bemis has shown that pupils solve problems stated in topical form, that is, grouped about some topic, such as earning money, etc., as accurately as they solve the same problems when stated in isolated form with other unrelated problems. The per cent of accuracy in solving topical problems was only .1% higher than for isolated problems. (Unpublished report.)

Baumgarten and Brueckner have shown that problems

[1] Monroe, W. S., "How Pupils Solve Problems" in Arithmetic, Bulletin No. 44, *Bureau of Educational Research* (Urbana, Ill., Univ. of Ill., 1928), p. 19.

which contain much related material to "enrich" them which is unessential in solving the problem are solved with much less accuracy than problems which are stated concisely and clearly. Apparently the reading difficulties introduced by the greater length of the problem offset any gain there may be in comprehension of the problem because of the greater richness in detail. (Unpublished report). Wheat[1] found very little if any difference in the difficulty of simple and enriched two-step problems.

Problems for Study, Reports, and Discussions

1. Select standard reading tests that would aid in the diagnosis of causes of difficulty in problem solving.
2. What are the functions of arithmetic as presented in this chapter?
3. Why might it be said that a typical verbal problem as stated in a textbook is in fact not a "problem" in the strict psychological sense?
4. In what way may practice in problem solving aid the pupil in the application of processes in life activities?
5. How could you find out if your pupils were inferior in arithmetic vocabulary?
6. Describe techniques that may be used to diagnose pupil difficulties in problem solving.
7. What per cent of failure to solve problems correctly is due to faulty computations?
8. What reasons can you give for the absurd answers pupils often give in solving problems?
9. Suggest local school situations in which problematic situations arise requiring the use of number processes.
10. Why is a rich background of number concepts essential to the development of ability to "think while reading"?
11. Test your class with a standard problem scale, graph

[1] H. W. Wheat, "The Relative Merits of Conventional and Imaginative Types of Problems," *Teachers College Contributions to Education*, No. 359 (New York: Columbia University, 1929).

the test scores, and analyze the work to determine the causes of the failure to solve the problems. Suggest a series of remedial exercises to overcome the deficiencies.

12. Make a careful case study of some pupil who is considerably below the standard in problem solving.

13. What faulty types of problems are described in this chapter?

14. Examine problems in some new textbook to discover the extent to which the problems are pitched to the level of the pupils' experiences and are likely to appeal to them as worth while.

15. Do you think that the discussion of such topics as the evolution of money has a place in the arithmetic curriculum? Why?

16. Do you think that teachers are able to prepare good original problems? To what extent should the problems used in the arithmetic period be taken from a basal textbook?

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Chapter VIII

THE IMPROVEMENT OF ABILITY TO SOLVE PROBLEMS

1. Experimental Evidence as to the Effectiveness of Remedial Work in Problem Solving.

Several important investigations have been made to determine the effectiveness of remedial instruction in problem solving. Lutes[1] showed that a marked increase in scores on problem scales resulted from practice on examples involving computations similar to those found in the problems on the scales. The reason for this increase in scores was probably increased accuracy and efficiency in computation.

Miss Wilson[2] reports an experiment in which the Buckingham Scale for Problems in Arithmetic was given to a group of pupils in grades 4 to 8. For a period of weeks, the teacher devoted ten minutes each day, three times a week, to the teaching of problems. The following types of exercises were used:

1. Estimating answers and judging absurdities. One exercise asked pupils to judge whether answers were reasonable or absurd. These problems were all selected from the Buckingham scale and the absurd answers were actually given by pupils taking the test.

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[1] O. S. Lutes, "An Evaluation of Three Techniques for Improving Ability to Solve Problems," *University Monographs in Education*, No. 6 (Iowa City, Iowa, 1926).

[2] E. Wilson, "Improving the Ability to Solve Arithmetic Problems," *Elementary School Journal*, Vol. 22, pp. 380–86.

2. Another exercise asked pupils to restate sentences using other words then those which were underlined. For example, in the following statement, "I can buy pencils *at the rate* of two for five cents" the pupils were asked to substitute words for the three words "at the rate." This was done in order to find out how much of the difficulty is due to such arithmetical phrases as "at the rate of," "total," and "average." An attempt was made also to discover how many of the difficulties were due to incomplete comprehension or to the skipping of words which are crucial to the solution of the problem correctly.

3. A third exercise asked pupils to read the problem and to indicate the processes necessary to its solution.

At the end of the period, the results of another form of the Buckingham Test showed a decided improvement in ability to solve problems and indicated that the special work had raised the average of the classes, which used the exercises, considerably more than the increase made by a control group which had no remedial work. This would indicate that there is a real need for vital problems that will present vivid pictures to the children and leave little doubt as to the meaning of the problem and what is wanted. Miss Wilson's experiment also shows the value of a systematic teaching program as a means of increasing the ability to solve problems.

Newcomb[1] made an investigation of the value of teaching pupils effective methods of attack in problem solving. He states that psychological investigations have shown that many difficulties in problem solving are due to the use of faulty methods of attack. He describes the following method as one that may be used effectively in the solution of problems:

[1] R. S. Newcomb, "Teaching Pupils How to Solve Problems in Arithmetic," *Elementary School Journal*, Vol. 23, pp. 183–89.

1. Reading the problem over carefully and thoughtfully before attempting a solution.
2. Look up the meaning of any unfamiliar word.
3. Analyze and arrange in an orderly manner the data given and determine the precise data required.
4. Select in the proper order the various processes necessary to effect a solution, deciding beforehand a reasonable result to expect.
5. Carefully check or evaluate the final result secured.

In an experiment very similar to the one reported by Miss Wilson, Newcomb found that when pupils were taught to use these methods, the results of Stone's Reasoning Test given before the method was applied and after a twenty days' training program showed that great improvement had been made. This would indicate very clearly that if a pupil attacks a problem in a random manner or uses faulty procedures, the difficulty, misunderstandings, lack of interest, and wrong solutions which are the inevitable result can be obviated to a considerable extent by teaching the pupils a definite method of attack.

P. R. Stevenson[1] conducted a study to determine the difficulties which pupils had in solving problems and to devise remedial instruction suited to individual cases. Tests were given to the pupils at the beginning of the experiment to find the individual disabilities and again at the conclusion of the experiment to find what was gained from the twelve weeks of remedial instruction. The test used was the "Buckingham Scale for Problems in Arithmetic" already referred to in this chapter. The remedial instruction covered twelve weeks and was repre-

[1] P. R. Stevenson, *Report of a Nation-Wide Testing Survey in Problem Solving* (Bloomington, Illinois: Public School Publishing Company, 1926).

sented by three periods of fifteen minutes each week. The pupils were taught to read and analyze problems, and to state clearly what facts were given. They were taught to write what questions were asked and to think what process or processes would be used and what the approximate answer would be. They were given practice in solving problems without numbers such as, "If you knew the amount of cloth needed to make a dress how could you find the amount necessary for three dresses?"

The pupils were also given a large variety of problems from actual life situations and were given special exercises in the meaning of difficult words in the problems. The test given at the end of the experimental period showed a growth of two years in some cases. Another important conclusion was that the dull pupils and normal pupils showed greater improvement than the bright pupils (those whose intelligence quotient was above 110) and that of the three groups, the dull pupils made the greatest gain.

Two studies of vocabulary difficulties have been reported, one by Monmouth[1] and others, and the other by Turner.[2] The results of the first study show that "vocabulary difficulties are real and not imaginary. Language of arithmetic books and problems should be clear, simple, and attractive. Terms and phrases too difficult and technical for pupils who are expected to solve them should not be used."

Turner's study resulted in some interesting conclusions regarding word difficulties. It indicates that authors use some words in lower grades but drop them in higher

[1] R. Monmouth, *et al.*, in Educational Research Bulletin, Vol. 3, No. 18 (Columbus, Ohio: Ohio State University, 1924), pp. 479–81.
[2] J. M. Turner, in Educational Research Bulletin, Vol, 3, No. 18 (Columbus, Ohio: Ohio State University, 1924), pp. 482–85.

grades; that when a new word appeared it was repeated many times on a few pages and then never used again; that many new words were added with each book, etc. This would tend to show that many of the difficulties that pupils have in problem solving are due to causes for which textbook writers and authors are responsible.

An experiment dealing with methods to be used in problem solving has been reported by Washburne and Osborne.[1] After a period of preliminary investigation, the experimental work centered about a comparison of three methods of teaching problem solving:

1. Simply giving the child many practical problems to solve.

2. Training the child to analyze each problem according to a definite technique which was prescribed.

3. Training the child to see the analogy between the more difficult written problems and simple oral problems of the same type.

A detailed description of the three methods is given in the report. While the work of the committee which coöperated in this experiment extended over a period of two years, the three methods of teaching were compared on the basis of a training period of only six weeks in length. The outstanding conclusion and the recommendations are stated as follows:

> Training in the seeing of analogies appears to be equal or slightly superior to training in formal analysis for the superior half of the children; analysis appears to be decidedly superior to analogy for the lower half; but merely giving many problems, without any special technique of analysis, or the seeing of analogies, appears to be decidedly the most effective method of all.

[1] C. W. Washburne and R. Osborne, "Solving Arithmetic Problems," *Elementary School Journal*, Vol, 25, pp. 219–26, 296–305.

The general recommendations growing out of the investigation are as follows: Problems should be so constructed as to present real situations familiar to the child. Children should be given many such problems to solve without special training in any generalized, formal technique of analyzing problems. *Concentration on practice in solving practical problems will yield gratifying results.*

A second experiment reported by Washburne[1] was concerned with the comparison of the efficiency of two methods of teaching pupils to apply the mechanics of arithmetic to the solution of problems. Each of two groups of children of approximately equal ability was taught by a different method, described by Washburne as follows:

> Group 1 was to be taught a number process through the use of verbal problems and with constant application to problems.
>
> Group 2 was to be taught the same number process without regard to problems or concrete situations until the mechanics were fairly well mastered, and then it was to concentrate on problem-solving. Both groups were to be taught by the same teacher. When one group was receiving oral instruction, the other group was to be out of the room.

Washburne's description of the method used in teaching Group 1 the "problem method" will make this method clear:

> The problem method as used in the experiment is based on the theory that pupils will learn a mathematical process more efficiently if they are faced with a problem which requires the use of the process. To have a stimulating effect on the pupils, a problem must be real, involving a situation that is familiar and demanding a solution which

[1] C. W. Washburne, "Comparison of Two Methods of Teaching Pupils to Apply the Mechanics of Arithmetic to the Solution of Problems," *Elementary School Journal*, Vol. 27, pp. 758–67.

has value in itself—in other words, a problem which the pupils will want to learn how to solve. The theory further postulates that the pupil desiring to solve such a problem not only will learn the process more easily but will be more willing to practice on other problems and drill materials in order to perfect himself in the solution of problems of this practical and appealing sort.

The method employed in the case of Group 2 (Method 2) consisted in simply plunging into the new process without assigning any reason and without giving any verbal problems. The mechanics of the process (or the facts) were taught independently. When, at the end of four weeks these mechanics were fairly well mastered, there was a two-week drive on problems, the same problems being used as were used by Group 1.

At the end of the six-week period both groups were given a problem-solving test in the newly learned process and a test in the mechanics of the process.

The results of the experiment and the significant conclusions of the committee were summarized in the following statement, quoted directly from the report submitted by the committee:

The Committee of Seven concludes, therefore, that teaching the mechanics of arithmetic—facts and processes—by themselves first, and then applying them to the solution of practical problems, does not lead to difficulty in making practical application of the mechanics to the solution of problems. A combination of thorough training in the mechanics of arithmetic and thorough training in the use of these mechanics in solving practical problems produces good results. This is true regardless of whether or not the mechanics are introduced through problems and constantly used in practical problems while they are being learned. The mechanics of arithmetic may be taught thoroughly and then applied to practical problems or the two types of teaching may be intimately related throughout the teaching process with equal efficacy.

2. SUGGESTED REMEDIAL PROGRAM IN PROBLEM SOLVING.

The experiments of Lutes, Wilson, Newcomb, Stevenson, Washburne, and others that have been briefly summarized show the value of a systematic teaching procedure as a means of increasing the ability of pupils to solve verbal problems. One of the important steps in improving the ability to solve verbal problems is to work for increased skill and accuracy in computation. To this end it is essential that well organized practice exercises in the various processes be used. Diagnostic tests and remedial exercises should be used to discover and remove specific sources of difficulty. Special stress should be placed on exercises in accurate reading in all subjects to eliminate another potent source of error, namely, inaccurate reading.

The essential elements of a training program for increasing ability in solving problems are well presented in the following quotation from a report of a special study by Stevenson,[1] which describes a procedure that has been successfully used after tests in problem solving have been given. Many of the types of exercises he suggests are described in Chapter 9 and were used in the experiments on improving problem solving that have been reported.

> All sane testing programs should involve the following procedure: (1) give tests, (2) locate individual difficulties, (3) apply remedial instruction, (4) give tests again to see if remedial instruction was effective. The following project has been tried out several times and has proved successful in increasing pupils' ability to solve problems.

[1] P. R. Stevenson, *Report of a Nation-Wide Testing Survey in Problem Solving* (Bloomington, Illinois: Public School Publishing Company, 1926).

The remedial instruction is planned to cover a period of twelve weeks. All the work is to be done in the recitation period. Three fifteen-minute periods each week are to be devoted to special remedial work. This makes a total of nine hours.

First, second, and third weeks.—During these three weeks the pupils should be taught to read and analyze problems. The pupils should be directed to open their books to a page containing a list of problems. Before solving a problem, drill should be given in finding what facts are given in the problem, what is asked, the process or different processes which should be used in solving the problem, and the answer in round numbers. (Some of the problems should be worked to see if the estimated answers are approximately correct.)

A list of problems should also be written on the board and pupils instructed to answer the above types of questions either orally or in writing. The teacher should study the results of the analysis test, and see that each pupil gets practice in the exercises upon which he failed. Pupils having the same difficulties may be given instruction in groups.

Fourth, fifth, and sixth weeks.—The pupils should work a large variety of problems during these three weeks. These problems should contain data from actual life situations.

The teacher should ask the pupils to submit lists of problems in addition, subtraction, multiplication, and division. These problems should be similar to those which the pupils and their parents have occasion to solve.

The teacher should try to arouse pupils' interest in solving different types of problems.

Seventh, eighth, and ninth weeks.—During this time have pupils solve problems without the use of numbers. These problems will, of necessity, have to be made up by the teacher and should be adapted to each grade. The following examples will furnish suggestions:

1. If you knew the amount of cloth needed to make a dress, how could you find the amount necessary for three dresses?

2. If you knew the cost of one dozen oranges, how would you find the cost of six oranges?
3. If you knew the height of each pupil in your class, how could you find the average height?
4. If you knew the amount paid for a house and the rate of commission, how could you find the commission?
5. If you knew the present population of your town and the population five years ago, how could you find the amount of gain? The per cent of gain?

Tenth, eleventh, and twelfth weeks.—During these weeks teach pupils to read problems and have them study their vocabulary. Devices which will be of assistance here are:

1. Have pupils state problems in their own words. Have each problem stated in as many different ways as possible.
2. In various problems, study words which might cause difficulty. Have the children explain the meanings of or define such words. Some words with which children have difficulty are: area, average, dealer, retail, commission, salary, rent, broker, wages, merchant, debt, expenses, acre, income, profit, loss, and insurance. Whenever possible, concrete explanations should be made by the teacher or the pupils.

Problems for Study, Reports, and Discussions

1. Summarize the results of experiments to determine the possibility of improving the ability of pupils to solve problems.

2. What types of exercises in problem solving were used in these experiments?

3. Outline a teaching procedure that you think would result in improvement in problem solving.

4. Examine textbooks to discover the problem solving helps they include.

5. Do you think that enough experimental work has been done to determine the relative merits of different methods of improving problem solving?

6. What questions do the facts presented in this chapter raise in your mind?

7. How will your teaching of problem solving be affected by the data here presented?

8. Does the chapter contain data as to the value of original problems prepared by pupils? What is the value of such problems?

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CHAPTER IX

DIAGNOSTIC AND REMEDIAL EXERCISES IN PROBLEM SOLVING

The chief causes of difficulty in problem solving revealed by the discussions of investigations summarized in the preceding chapter are:

1. Lack of ability to perform the necessary computations accurately or to select the operation needed.

2. Lack of systematic method of attack in solving a problem.

3. Careless reading or lack of vocabulary.

4. Lack of knowledge of essential facts, data, or principles involved.

5. Failure to complete the problem.

6. Failure to comprehend the problem in whole or in part.

1. NECESSITY OF IMPROVING ACCURACY IN COMPUTATION.

Investigations have shown that from 20 to 40 per cent of all errors in problem solving are due to errors in computation. Lutes[1] has shown that systematic preliminary practice on examples involving computations similar to those used in the examples in the problem test results in a marked increase in the scores made by pupils. "Improvement in computational ability does increase ability to solve verbal problems; whether it increases

[1] O. S. Lutes, *op. cit.*

ability to do arithmetic reasoning cannot be stated from the results of this study; it does improve ability to earn scores on tests of ability to solve verbal problems, which is the school's definition of arithmetic reasoning."

Lutes found that practice in computation was more effective than two methods he labeled "the method of choosing operations" and "the method of choosing solutions."

Winch,[1] on the other hand, reports an experiment in which it was found that there was no transfer from general practice in computation to ability to solve reasoning problems. Evidently problems and examples used during practice periods ought to deal at least in part with similar processes so as to reduce the amount of inaccuracy in problem solving because of difficulty in computation, or inability to perform the necessary operation.

It is known that large numbers in problems tend to confuse pupils. When simpler numbers are introduced in the problem it is often easily solved by pupils. Large numbers are often entirely beyond the comprehension of pupils. The possibility of error is also greater in computations with large numbers. The use of smaller numbers in problems will therefore tend to increase the ease of comprehension of problems and to decrease the amount of inaccuracy in computation.

2. Informal Types of Exercises in Problem Solving.

Special types of informal exercises may be used to locate pupil difficulties in solving verbal problems and to develop essential skills.

These exercises are similar in structure to the various kinds of objective devices that are being widely employed

[1] W. H. Winch, "Further Work on Numerical Accuracy in School Children," *Journal of Educational Psychology*, Vol. 2, pp. 262–71.

to improve the written examination. Illustrations of the most important of these are as follows:

1. Multiple choice:
   (*a*) In one foot there are 2, 8, 12, 16, inches.
   (*b*) Draw a line around the number below which is most likely to be the correct product of 18 × 27:
   250, 366, 486, 495

2. Completion:
   (*a*) The answer of an .... example is called the sum.
   (*b*) $\frac{2}{3}$ and $\frac{1}{3}$ are called .... fractions.

3. True false:
   Mark the sentences that are true (*c*). Mark those that are false (*x*).
   (*a*) In a yard there are 3 feet.
   (*b*) The formula for finding the area of a circle is $2\pi r$.
   (*c*) Interest is the same as amount.
   (*d*) Six per cent is equal to .06.

4. Simple recall:
   (*a*) How many square rods are there in an acre? ....
   (*b*) What is the formula for finding the area of a rectangle? ....
   (*c*) The answer in a multiplication example is called the ....

5. Recognition:
   What process would you use in solving each of these problems? Write (*a*) for addition, (s) for subtraction, (*m*) for multiplication, and (*d*) for division.
   (1) What is the cost of two apples at 5 cents each? ....
   (2) What is the cost of butter a pound if 6 pounds cost \$3.24? ....

6. Yes—No:
   Underscore the answer you believe to be correct.
   (*a*) Is a mile equal to 160 rods? Yes—No.
   (*b*) Is an ounce less than a pound? Yes—No.
   (*c*) Is the formula for interest, $i = prt$? Yes—No.

7. Matching exercises:

Before each item at the left write the number of the correct formula from the list at the right.

.... Area of a square	1. $P = 4s$
.... Volume of a rectangular solid	2. $V = lwh$
.... Perimeter of a square	3. $A = S^2$

8. Selection exercises:

Draw a line around the numbers of the statements below the problem which are not true.

(*a*) Jack bought 7 apples at 3 cents apiece. How much change should he receive from a quarter?

1. Jack bought 7 apples.
2. Jack paid 5 cents apiece for the apples.
3. He gave the clerk 15 cents.
4. He received no change.
5. Jack sold 7 apples.

Exercises based on the principles involved in the construction of the new type of objective examination can be prepared to give practice on such important elements in problem solving as the following:

1. Ability to name processes in solving problems.
2. Ability to select the essential facts.
3. Ability to estimate answers to problems.
4. Ability to check answers.
5. Ability to select processes in solving problems containing two or more steps.
6. Knowledge of vocabulary needed in problem solving.
7. Knowledge of essential concepts and facts.
8. Ability to check true and false statements in review exercises.
9. Ability to assemble essential data.
10. Ability to read carefully and exactly.
11. Ability to attack the solution of a problem systematically.
12. Ability to apply processes in local situations.
13. Map reading exercises involving quantitative concepts.
14. Interpretation of tables and graphs.

15. Uses of index, reference books, etc.
16. Making analogies.
17. Formulating problems involving given facts or processes.
18. Formulating problems involving easier numbers.
19. Answers in specific questions about problems.

Excellent descriptions of the new types of examinations and discussions of their construction, reliability, and utility are given in G. M. Ruch's *Improvement of the Written Examination,* by Scott Foresman and Company, and in D. G. Paterson's *Constructing New Type Examinations,* by the World Book Company. Modern arithmetic textbooks contain illustrations of how these new types of exercises can be applied to developing the specific abilities involved in problem solving.

3. Improving Comprehension of Meaning of Problems.

Failure to comprehend the problem in whole or in part is evidenced by failure to work it at all, or by the use of faulty processes in solving it. The reasons for the difficulty, if it is a general characteristic of the work, may be lack of native capacity or lack of background. The problem may be outside the range of the experiences of the pupil or the vocabulary may present unusual difficulties. The pupil may be unable to select the essential elements in a problem. He may be unable to determine the outcome of the problem or he may be unable to select the correct process or processes to use in solving the problem; the relationships involved may be too difficult. It is, therefore, important that the problems that are given to the pupils to solve must be within the range of their experiences. Systematic work must be done to

develop the arithmetic vocabulary of pupils. Much depends on the type of work done during the arithmetic period in the primary grades. Important basic concepts should be established in these grades by means of concrete illustrations. Meaningful applications of number in a wide variety of situations should be stressed. Directed practice in careful reading, such as following directions, selecting important ideas, restating the meaning of a sentence in the words of the pupils, and similar exercises, should form a part of the training in arithmetic. The ability of the pupil to *solve* verbal problems depends largely on his ability to *read* them intelligently.

4. Vocabulary Exercises.

The lack of an adequate vocabulary has been shown by Chase[1] who undertook an investigation to show what certain words frequently found in arithmetic texts mean to children. Her test exercise included 47 words, preceded by the following instructions:

1. Put *w* beside each word that tells what a man's work is.
2. Put *m* beside each word about money.
3. Put *l* beside each word that might be used about land.
4. Put *i* beside each word that is the name of something to put things in.

The list of words with the per cent of pupils in the fourth and fifth grades who did *not* indicate the correct meaning follows:

	Grade IV	Grade V		Grade IV	Grade V
acre	56	32	basin	26	20
area	91	36	basket	13	8
attend	4	4	bin	43	20
barrel	30	32	broker	56	45

[1] Sara E. Chase, "Waste in Arithmetic," *Teachers College Record*, Vol. 18, pp. 360–70.

	Grade IV	Grade V		Grade IV	Grade V
bucket..........	30	16	lot..............	9	12
building lot....	65	32	machinist	35	28
carpenter........	26	8	mason...........	78	36
cashier..........	52	20	merchant........	45	24
cistern..........	100	88	miller...........	26	12
coins............	26	16	nickel...........	13	16
collect..........	17	8	owe.............	91	60
commission......	100	68	pasture.........	52	32
customer........	35	44	poultry..........	40	36
dealer...........	35	28	profit...........	78	45
debts...........	91	32	real estate.......	100	68
earn............	60	36	rent.............	56	28
excavate........	4	4	retail...........	30	16
expenses........	65	20	salary...........	40	20
fares............	65	60	schedule........	17	4
field............	9	8	tailor...........	13	4
gardener........	17	8	tank.............	65	36
income..........	96	41	teamster........	88	40
insurance........	91	42	wages...........	70	28
jars.............	40	16			

In commenting on this table, Morton[1] says:

Unfortunately, no data are given for the third grade. We can infer, however, that the per cents for the third grade would not be higher than those given for grade four. It will be noted that the word, dealer, which we found in a problem, and which Thorndike lists in the fourth thousand according to frequency of occurrence, was unknown to 35 per cent of the pupils of the fourth grade.

Similar vocabulary exercises should be used to insure the development of the arithmetic vocabulary. Another typical vocabulary exercise follows:

An Arithmetic Vocabulary Test[2]

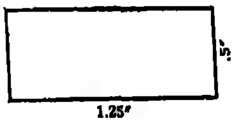
In the fifth grade you learned many words frequently

[1] Morton, R. L., *Teaching Arithmetic in the Primary Grades* (N. Y.: Silver Burdett and Company, 1927), p. 201.

[2] The illustrative types of reading exercises found in this chapter are typical of the many kinds of problem-solving helps found in *Triangle Arithmetics*, Grades 3 to 8, and in *Diagnostic Tests and Practice Exercises*, Grades 3 to 8, published by The John C. Winston Company.

used in arithmetic. This test will help you to find how many of the words you remember.

1. The answer in an addition example is called the ....
2. The answer in a multiplication example is called the ....
3. The answer in a division example is called the ....
4. $\frac{1}{2}$ is called a .... fraction.
5. $\frac{7}{4}$ is called an .... fraction.
6. $3\frac{3}{4}$ is a .... number.
7. $\frac{1}{4}$ and $\frac{3}{4}$ are a pair of .... fractions.
8. $\frac{2}{3}$ and $\frac{1}{2}$ are a pair of .... fractions.
9. In the fraction $\frac{4}{5}$, 4 is the ....
10. In the fraction $\frac{7}{8}$, 8 is the ....
11. 15 is a .... denominator of $\frac{1}{3}$ and $\frac{1}{5}$.
12. $\frac{3}{4}$ is the .... form of $\frac{4}{3}$.
13. Changing $\frac{1}{4}$ to $\frac{2}{8}$ is called ....
14. Changing $\frac{8}{12}$ to $\frac{2}{3}$ is called ....
15. The numbers 2 and 3 in $\frac{2}{3}$ are called the .... of the fraction.
16. A square inch is a unit of ....
17. The figure at the right is a ....



18. In the number, 7.5, the (.) is called the ....
19. The number, .5, is a .... fraction.
20. A cubic inch is a unit of measure of ....
21. The distance around a flat surface is its ....

5. Exercises in Careful Reading.

The following exercises can be used to determine the ability of the pupil to grasp the meaning of the problem and to comprehend the facts included in it. They also give the pupil practice in careful reading.

(A) Reading Exercise in Problem Solving

Under each of the following problems there are statements. Tell which of them are true and which are false.

1. *Eggs cost 48 cents a dozen. Mary buys 2 dozen. How much change should she receive from a dollar?*

a. Mary buys 3 dozen eggs.
b. The eggs cost 48 cents a dozen.
c. Divide to find the cost of the eggs.
d. Mary spent less than a dollar for the eggs.
e. You are to find what Mary spent for the eggs.

2. *Arthur spends 20 cents a day for lunch and 16 cents a day for street-car fare. How much does he spend for car fare and lunches in five days?*

a. Each day Arthur spends 36 cents for car fare and lunch.
b. Arthur spends 16 cents for car fare in five days.
c. Arthur spends 20 cents a day for lunch.
d. In five days Arthur spends $5.00 for lunches and car fare.
e. Arthur spends more for street car fare than for lunch.

3. *Mary's mother bought 6 yards of cloth at 45 cents a yard for two dresses for Mary. Find the cost of the cloth for one dress.*

a. Mary's mother bought 45 yards of cloth.
b. She made two dresses with the cloth.
c. The cloth cost 45 cents a yard.
d. She needed three yards of cloth for one dress.
e. I am to find the cost of the cloth for the two dresses.

4. *What is the area of a field that is $\frac{7}{8}$ mile long and $\frac{3}{4}$ mile wide?*

a. Area means surface.
b. Area is the same as perimeter.
c. The field is longer than it is wide.
d. $\frac{7}{8} \times \frac{3}{4}$ will give the area.

(B) Problems With Facts Missing

In each of the following problems there is some fact missing which must be known before the problem can be solved. What fact is missing in each of the problems?

1. Mary bought bread at 12 cents a loaf. She gave the clerk 50 cents. What change should she receive?

2. Harry sold papers at 3 cents each, for which he paid 2 cents apiece. How much was his gain on all of the papers that he sold?

3. Jack had 5 cents more than Harry. How much did Harry have?

4. Alice sold flowers for 10 cents a bunch. How much did she receive for all of the flowers she sold?

5. Mary received $2.40 for taking care of Mrs. Jackson's baby. How much was she paid an hour?

6. Harry and Robert bought some marbles. Harry took 20 of them and Robert took the rest. How many did Robert take?

7. Helen had a garden, 40 feet long. What would it cost to build a fence around the garden if wire fencing cost 15 cents a foot?

8. In the fourth-grade room there were six rows of seats. How many seats were there in the room?

9. Harry paid 30 cents for some pencils. How much did he pay for each pencil?

10. Helen had read 60 pages in her new book. How many more pages does she still have left to read?

11. Alice had a savings bank with some money in it. She put in 50 cents more on Monday and 15 cents on Wednesday. How much did she then have in the bank?

(C) Problems Without Numbers

1. If you know the number of miles an automobile is going an hour, how would you find the number of miles it will travel in 8 hours at this rate?

2. If you know the price of sugar a pound, how will you find the number of pounds you can buy for a dollar?

3. If you know the amount a man pays for the rent of his house for one month, how will you find how much rent he pays for a year?

4. If you know how many miles an automobile can travel in one hour, how will you find the number of hours it will take it to travel 2,000 miles?

6. Exercises to Determine Knowledge of Essential Data and Principles.

Suitable exercises to determine the pupil's knowledge of essential data and principles are as follows:

A. Test in Values of Units of Measure

Write the number that should be in each blank below.

1. 1 ft. = ... in.
2. 1 qt. = ... pt.
3. 1 bu. = ... pk.
4. 1 lb. = ... oz.
5. 1 hr. = ... min.
6. $1 = ... dimes.
7. 1 cwt. = ... lb.
8. 1 mi. = ... yd.
9. 1 bu. = ... qt.
10. 1 A. = ... sq. rd.
11. 1 mi. = ... ft.
12. 1 rd. = ... yd.
13. 24″ = ...′.
14. 1 wk. = ... da.
15. 1 pk. = ... qt.
16. $3\frac{1}{2}$′ = ...″.
17. 1 T. = ... lb.
18. 1 mi. = ... rd.
19. 1 dime = ... cents.
20. 1 gal. = ... qt.
21. 1 yd. = ... ft.
22. 1 yr. = ... mo.
23. 1 sq. ft. = ... sq. in.
24. 1 cu. ft. = ... cu. in.
25. 1 sq. yd. = ... sq. ft.
26. 1 min. = ... sec.
27. 1 doz. = ... things.
28. 1 quarter = ... nickels.
29. 1 ordinary yr. = ... da.
30. 1 leap yr. = ... da.
31. 1 rd. = ... ft.
32. 1 cu. yd. = ... cu. ft.

B. Test in Changing the Form of Units of Measure

1. 18 in. = ... ft. .... in.
2. 5 ft. = ... yd. .... ft.
3. 27 oz. = ... lb. .... oz.
4. 45 min. = ... hr.
5. 18 in. = ... yd.
6. 18 qt. = ... gal. .... qt.
7. 2,500 lb. = ... T. .... lb.
8. 10 pk. = ... bu. .... pk.
9. 7 pk. = ... bu.
10. 27 sq. ft. = ... sq. yd.
11. 1,728 cu. in. = ... cu. ft.
12. 880 yd. = ... mi.

C. Writing Tables of Measure

1. Write the table of square measure; of cubic measure.
2. Write the table of dry measure.
3. Write the table of liquid measure.

D. Using Letters for Words

You have learned to express rules in arithmetic with letters. Explain the meaning of each of the following:

1. $A = lw$	5. $V = lwh$	9. $C = \pi d$
2. $A = \frac{1}{2} ab$	6. $A = \pi r^2$	10. $V = bh$
3. $A = ab$	7. $d = 2r$	11. $A = \dfrac{a(b + b')}{2}$
4. $P = 2(l + w)$	8. $C = 2\pi r$	12. $A = s^2$

Explain the following statements about costs:

13. $c = np$ 14. $n = \dfrac{c}{p}$ 15. $p = \dfrac{c}{n}$

16. Net profit = Selling price − gross cost

Explain each of the following formulas about commissions:

17. $C = r \times$ Total sales

18. Rate of commission = $\dfrac{\text{Commission}}{\text{Total sales}}$

The necessity of looking up the meaning of important words in problems, the use of reference books, the index of the textbook and the appendix, to secure necessary information are stressed by exercises such as the following:

A. Looking Up Facts

In each of the following problems there is some important fact not stated which you must know before you can solve the problem. If you do not know the missing fact, look it up in your arithmetic, in the dictionary, in the newspaper, or wherever you think you can find it. After you have done this, solve the problem.

1. In the Olympic games a man vaulted 3.9 meters. How many feet and inches was this?

2. What is the value of 2,460 pounds of oats at $.85\frac{1}{2}$ a bushel?

3. In a game of football, the Carroll team made 3 touchdowns, and kicked the goal once; the Ripon team made 2

touchdowns, kicked the goal both times, and made 2 field goals. Which team won the game?

B. Problems With Numbers Missing

In the problems below some facts are missing. Supply the facts and then solve the problems. Be sure that the numbers you supply are reasonable.

1. A garden is .... feet long and .... feet wide. What is the distance around the garden?
2. The arithmetic class in the fifth grade begins at .... o'clock and ends at .... o'clock. How long is the class period?
3. Harry's father sold .... bushels of apples at .... a peck. How much did he receive for the apples?
4. Alice's mother bought a bushel of potatoes. When she reached home she weighed them on her scales and found that they weighed only .... pounds. How much less than a bushel did they weigh? (A bushel of potatoes weighs 60 pounds.)
5. Mary bought .... yards of cloth for a dress. The cloth cost .... What was the price of the cloth a yard?
6. Harry picked .... quarts of strawberries on Monday, ... quarts on Tuesday, ... quarts on Wednesday, and .... quarts on Thursday. He was paid .... cents a quart for picking them. How much did he earn in all?
7. Alice and John belonged to a pig club. When their pigs were weighed at the fair in the fall, Alice's pig weighed .... pounds and John's pig weighed .... pounds. How much more than John's pig did Alice's weigh?
8. The distance around the block in which Harry lives is .... yards. How much less than a mile is this?
9. Julia went to the store and bought .... quarts of milk at .... cents a quart and .... pounds of butter at .... cents a pound. How much did she pay the clerk?

7. Exercises for Giving the Pupil a Systematic Approach to Problem Solving.

The following exercises can be used to give the pupils

the steps in a systematic approach to the solution of problems and to determine weakness in any of the steps involved:

A. A STUDY EXERCISE ON PROBLEM SOLVING

1. If John sold 50 morning papers and 20 evening papers, how many papers did he sell that day?

a. What is the question you are to answer?

b. In order to find this, what must you know?

c. Does the problem tell you this?

d. How can you find the answer to the question the problem asks?

Study the following problem carefully:

2. If John delivers 50 papers in the morning and 20 papers in the evening, how many papers does he deliver in 3 days?

a. What is the question you are to answer?

b. Before you can find the number of papers a boy delivers in three days, what must you know?

c. Does the problem tell you the number of papers John delivers in one day?

d. You can see that this problem has one step that problem 1 did not have. How can you find the number of papers John delivers in one day?

e. When you know the number of papers John delivers in one day, how can you find the number he delivers in three days?

3. Mary bought 3 oranges at 5¢ apiece. She gave the clerk $.25. How much change should she receive?

a. What is the question you are to answer?

b. Before you can find the amount of change Mary should receive, what must you know?

c. Does the problem tell you how much Mary spent?

d. How can you find how much Mary spent?

e. When you know how much Mary spent, how can you find the amount of change Mary should receive?

B. LEARNING TO SOLVE PROBLEMS

Read each problem carefully. Be sure that you under-

stand all the words. Ask yourself questions *a*, *b*, and *c* about each problem. Then solve it and check your work.

a. What am I asked to find in the problem?

b. What things are told in the problem that will help me to find the answer?

c. What must I do to solve the problem?

1. Jack wishes a pair of skates. The price of the skates is $4.80. Jack can save $.80 a week. In how many weeks will he save enough money to buy the skates?

2. Fred took a trip with his father. When they started, the speedometer on the car read 8,569 miles. When they got home, it read 10,433 miles. How many miles did they travel on their trip?

3. Jennie spent $8.25 for cloth for a dress. She bought 3 yards. How much did the cloth cost a yard?

C. A Reading Exercise

I. The fifth- and sixth-grade children were to have a sleigh ride party. There were 46 fifth-grade pupils and 58 sixth-grade pupils. How many pupils were there in the party if they all went?

1. Which of the following facts are you asked to find?
   a. The number of pupils in the fifth grade.
   b. The number of pupils in the party if they all went.
   c. The number of pupils who did not go.
   d. The number of pupils who could ride in one sleigh.
2. Which of the following facts is given in the problem?
   a. The number of pupils who could not pay.
   b. The number of sleighs that would be needed.
   c. The number of pupils who could not go.
   d. The number of pupils in the two grades.
3. In which of the following ways would you work this problem?
   a. Add 58 and 46. *c*. Multiply 58 by 46.
   b. Subtract 46 from 58. *d*. Divide 58 by 46.
4. Which of the following answers is nearest to the *correct answer?*
   a. 94 pupils. *b*. 82 pupils. *c*. 106 pupils. *d*. 12 pupils.

D. Problems With Two Questions

In the problems below, you must find the answer to the first question before you can answer the second question.

1. Harry bought a tablet for 5¢ and a pencil for 5¢.
   a. How much did the pencil and the tablet cost?
   b. How much change would he receive from a quarter?

2. Mary practiced on her piano 30 minutes each morning except Sunday.
   a. How many minutes did she practice in a week?
   b. How many hours was this?

3. A milkman had two large cans of milk. One contained 18 quarts and the other 14 quarts.
   a. How many quarts of milk were there in the two cans?
   b. How many gallons were there?

4. John picked 15 quarts of blueberries on Monday and 18 quarts on Tuesday.
   a. How many quarts did he pick in the two days?
   b. If he sold them at 18¢ a quart, how much did he receive for the berries?

E. Solving Two-Step Problems

1. The children in grades 3 and 4 used 25 half-pint bottles of milk each morning. The children in grades 1 and 2 used 43 half-pint bottles. What was the cost of the milk at 3¢ a bottle?

   a. What am I to find?
   The cost of the milk.

   b. What facts are given?
   Grades 3 and 4 used 25 half-pint bottles.
   Grades 1 and 2 used 43 half-pint bottles.
   The cost a bottle is 3¢.

   c. How can I find the answer?
   The number of bottles used times the price of one bottle equals the cost of the milk.
   How many bottles were used? The fact is not given in the problem and must be found.

First Step:	*Second Step:*
25 bottles in grades 3 and 4 43 bottles in grades 1 and 2 68 bottles used	Finding the answer: $68 \times \$.03 = \$2.04$

d. Is the answer reasonable?

Answer questions *a*, *b*, *c*, and *d* about these two problems:

2. The children in a school wish to raise money to buy swings for the playground at a cost of \$85. They sold 258 tickets for a school fair at 10¢ apiece. How much must they still earn before they can pay for the swings?

3. One day Alice picked 15 bunches of flowers. She gave three of the bunches to her mother and sold the rest at 15 cents a bunch. How much did she receive for the flowers she sold?

8. Exercises to Teach the Pupil to Select the Correct Process or Processes.

Inability of the pupil to solve a problem is often due to his not being able to select the process or processes to be used in solving it. The following exercises can be used to determine the ability of the pupil to select processes to be used in solving one-step and two-step problems:

A. Naming Processes Used to Solve Problems

What process would you use to solve each of the following problems? Write *a* for add, *s* for subtract, *m* for multiply, and *d* for divide.

Example: How do you find the cost of 25 pounds of meat at 34 cents a pound? ......*m*......

1. How do you find the price of a pound of coffee if $5\frac{1}{2}$ pounds cost \$2.75? ....

2. How do you find the cost of $7\frac{1}{2}$ pounds of sugar at 8¢ a pound? ....

3. Mrs. Johnson needed 5 pounds of flour for bread. If she had only $3\frac{3}{4}$ pounds of flour, how many pounds more does she need? ....

4. Mr. Smith, a dairy farmer, sold 45 pounds of butter on Monday, 36 pounds on Tuesday, and 29 pounds on Wednesday. How many pounds of butter did he sell? ....

5. Mary bought 5 pounds of butter for $2.45. How do you find the price a pound? ....

6. In the Calhoun School there are 240 boys and 238 girls. How do you find the total number of children in the school? ....

7. Mr. Johnson sold 30 chickens that weighed in all 105 pounds. How do you find the average weight of the chickens? ....

8. A farmer sold 196 bushels of apples at $2.25 a bushel. How do you find how much he received for the apples? ....

9. How do you find the total weight of three boys, one weighing $84\frac{1}{2}$ pounds, one weighing $67\frac{3}{4}$ pounds, and one weighing $86\frac{3}{4}$ pounds?

B. Stating Processes to Use in Solving Two-Step Problems

On the line after each problem state the processes you would use to solve the problem.

1. Mr. and Mrs. Anderson with their three children took a ride on a street car. The fare for each was 7¢. Mr. Anderson gave the conductor a dollar bill. How much change did he receive?

A. .... and ....

2. Mr. Jones took 492 eggs to market. He sold them at 35¢ a dozen. How much did he receive for the eggs?

A. .... and ....

3. There are 8 rows of trees in an orchard and 17 trees in a row. Sixteen of the trees are cherry trees. The rest are apple trees. How many apple trees are there?

A. .... and ....

4. A truck loaded with coal weighs 6,840 pounds. The truck when empty weighs 2,840 pounds. How many tons of coal are in the load? (2000 pounds to a ton.)

A............ and ....

C. Problem Study: Finishing Problems

In the fourth grade the pupils play a game. One pupil writes two facts on the board and then puts a sign +, −, ×, or ÷ after them. This means that the pupils must make up a problem with those two facts. The question must be answered using the process the sign tells.

John wrote these facts: The price of butter is 47¢ a pound. 18 pounds. He made this sign: ×.

Alice wrote the problem this way: The price of butter is 47¢ a pound. Find the cost of 18 pounds of butter.

Make up a problem for each of the following pairs of facts, using the process the sign tells:

1. The price of coffee is 48¢ a pound. 4 pounds. ×
2. John has 28 marbles. Will has 24 marbles. +
3. Oranges cost 60¢ a dozen. One orange. ÷
4. A boy bought a cap for $1.50. He gave the clerk a five-dollar bill. −

9. Basal Types of Verbal Problems in Arithmetic.

The inability of the pupil to select the correct process by which to solve a problem may be due to one or more of the following reasons: the fact that he has not learned to recognize solution patterns involved in the problem; to lack of basic facts not given in the problem; to lack of knowledge of certain principles involved, and to other similar factors.

Ability to analyze situations, and then to select the correct method of solution depends to some extent on the provisions that have been made in the instructional material. Provision should be made to give the pupil

experience in solving problems involving the important basal types of solutions, found in arithmetic. An analysis of arithmetic situations shows that these basal types of solutions can be classified as eleven different kinds of patterns. Each type may occur in isolation in one-step problems, or in combination with others in two- or three-step problems. The eleven types and examples of each are as follows:

1. The addition problem. (One type.)

Example. Jack has 4 cents, Mary has 5 cents, and Alice has 7 cents. How many cents do they have in all?

2. Subtraction problem. (Three types.)

(*a*) Type 1. *Taking away.*

Example. Jack had 10 cents. He spent 5 cents. How much did he have left?

(*b*) Type 2. Comparing.

Example. Harry has 15 cents. Jane has 10 cents. How much more has Harry than Jane?

(*c*) Type 3. Increasing.

Example. Jack has 5 cents. How much more must he have before he can buy a 15-cent top?

3. Multiplication. (Two types.)

(*a*) Type 1. Told about one; asked about many.

Example. An apple costs 5 cents. How much do 5 apples cost?

(*b*) Type 2. Finding a part of a number.

Example 1. Find the cost of $\frac{3}{4}$ pound of butter at 40 cents a pound.

Example 2. How much is 15% of $400?

4. Division. (Five types.)

(*a*) Type 1. Told about many; asked about one.

Example. If 5 apples cost 15 cents, how much does one apple cost?

(*b*) Type 2. Finding a number of equal groups.

Example. How many apples at 5 cents each can I buy for 50 cents?

(*c*) Type 3. Finding what part one number is of another.

Example 1. What part of 50 is 25?

Example 2. At a sale I bought a $4.50 pair of skates for $1.50 less than the regular price. What was the per cent of savings?

(*d*) Type 4. Finding the whole with a part given.

Example 1. If $\frac{3}{4}$ pound of butter costs 30 cents how much does one pound cost?

Example 2. Frank received $3.00 interest at the end of the year on money deposited in his savings account. The bank pays 3% interest a year. How much money did Frank have on deposit in the bank?

(*e*) Type 5. Finding a missing factor.

Example. Find the width of a field containing 4,000 square rods if its length is 80 rods.

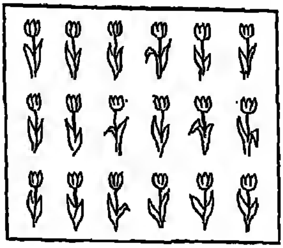
Experimental work at Waukesha, Wisconsin, by Banting and Merton has shown that marked growth, in ability to solve verbal problems, results from the use of reading exercises in problem solving. In these exercises the pupils are taught to recognize the various types of situations and solutions outlined above. The exercises of this type that follow, are typical of the types of problem-solving exercises that have been developed during the experimental work.

(*a*) The following exercise[1] suggests how pupils may be taught to draw pictures to represent situations which will help them to solve the problem:

[1] The three exercises given are taken from *Triangle Arithmetics*, Book I, Part I.

Helen's teacher gave her the following problem. Helen drew a picture of the multiplication problem in order to understand it better:

Grace has a flower garden. She planted 5 tulips in each row. How many tulips did she plant in 3 rows?



This is how Helen drew the picture on her paper:

She drew 5 tulips in a row.

She drew 3 rows of tulips.

She saw that Grace has planted 3 times 5 tulips, so she multiplied to find 3 × 5 tulips. She then wrote on her paper what you see at the right.

$$\begin{array}{r} 5 \text{ tulips} \\ \times 3 \phantom{\text{ tulips}} \\ \hline 15 \text{ tulips} \end{array}$$

PROBLEMS FOR WHICH YOU MAY DRAW PICTURES

Draw pictures on your paper to show these problems:

1. There are 12 eggs in one dozen. How many eggs are there in 2 dozen?

2. Donald planted 8 tomato plants in each row. How many plants did he put in 3 rows?

3. The children were taken to the woods in 3 automobiles. Six children rode in each automobile. How many children went to the picnic in automobiles?

4. Each of three girls has 4 dolls. How many dolls have they in all?

5. Otto had 2 nickels changed into pennies. How many pennies did he receive?

(*b*) Pupils may be taught to recognize certain verbal cues which indicate the solution to use. The following exercises suggest the method of approach:

Some Helps to Remember

Here are some ways in which you are sometimes asked to *compare* by subtraction:

How much *more than* one is the other?
How much *less than* one is the other?
How much *older than* one is the other?
How much *younger than* one is the other?
How much *longer than* one is the other?
How much *shorter than* one is the other?
How much *farther than* one is the other?
What is the *difference between* the two?
How *many more* are needed?

When a problem tells you how many are in one group, and asks you to find how many are in more than one of these groups, you may add or multiply. Why is it better to multiply than to add?

For example: There are 11 players on one football team. How many players are there on 2 teams?

When a problem tells you the price of one article and asks you to find the cost of more than one of these articles, you may add or multiply. Why is it better to multiply than to add?

For example: At the picnic ham sandwiches sold for 6 cents each. Roger bought 3 sandwiches. How much did he pay for them?

Problems to Study

Study the following multiplication problems. Give reasons why you know they are multiplication problems. Then work each problem on your paper.

1. A ride on the merry-go-round costs 5 cents. Harry had 3 rides. How much did he pay for the 3 rides?

2. There are 12 oranges in a dozen. Ann's mother bought 3 dozen. How many oranges did she buy?

(*c*) The following exercise may be used to give pupils practice in telling why a certain process is used in solving a problem.

Looking for Helps in Problem Solving

In these three problems, how can you tell that you must add?

1. Mary had 4 black chicks, 3 white chicks, and 4 brown chicks. How many chicks did she have in all?

2. Jack had 5 marbles and Harry had 10 marbles. How many marbles did both boys have?

3. From one nest Mary gathered 5 eggs, from another nest 8 eggs, and from another nest 9 eggs. How many eggs did Mary gather in all?

In these three problems, how can you tell that you must subtract?

1. Jack had 10 cents and Mary had 7 cents. How much less money did Mary have than Jack?

2. Jack is 56 inches tall and Mary is 61 inches tall. How much taller than Jack is Mary?

3. Mary wishes to buy a game that costs 25 cents. She has 20 cents. How much more does she need?

In these two problems, how can you tell that you must multiply?

1. Find the cost of 4 apples at 5 cents apiece.

2. Helen sold 6 bouquets of flowers at 19 cents each. How much did she receive for the flowers?

In these three problems, how can you tell that you must divide?

1. How many 4's are there in 24?

2. In a classroom there are 36 seats. There are 6 rows of seats. How many seats are there in each row?

3. Mary had an 18-inch ribbon. She cut it into 3 equal parts. How long was each part?

10. Exercises Designed to Determine the Ability of the Pupil to Estimate Answers to Problems.

Exercises specially designed to determine the ability of the pupil to estimate answers to problems and examples are as follows:

A. Estimating Answers

After each of the following examples, three numbers are written. First select the number which you think is about the correct answer; second, work the example; third, compare your estimate with the correct answer. How many of your estimates were correct?

1. $75 + 86 + 97 = 300, 200, 250.$
2. $\$37.96 - \$28.97 = \$10, \$9, \$8.$
3. $25 \times 48 = 800, 1{,}200, 1{,}600.$
4. $\$56.25 \div 25 = \$3.20, \$8.15, \$2.20.$
5. $3\frac{1}{2} + 2\frac{1}{4} + 7\frac{1}{2} = 12\frac{1}{4}, 13\frac{1}{4}, 14\frac{1}{4}.$
6. $374\frac{1}{8} - 356\frac{1}{4} = 30, 40, 50.$
7. $5\frac{1}{2} \times 36 = 150, 200, 300.$
8. $3\frac{1}{2} \times 6\frac{1}{7} = 18, 23, 36.$
9. $9 \div \frac{2}{3} = 10, 14, 22.$
10. $2\frac{1}{2} \div 5 = 2, \frac{1}{2}, 3\frac{1}{4}.$

B. Practice in Estimating Answers

Accurate arithmetic workers estimate the answer to a problem before doing the work. They check their answer by comparing it with their estimate. This prevents many foolish errors. Under each of the following problems are three estimates. First read each problem and estimate the answer. Then work the problems. In how many problems were your estimates the same as the correct answers?

1. The winner of a 500-mile automobile race traveled this distance in $5\frac{3}{4}$ hours. What was his average speed an hour?

a. 75 miles *b.* 90 miles *c.* 110 miles

2. In the Calhoun School there were 28 classrooms. There were 43 desks in each room on the average. How many desks were there in the school?

a. 500 *b.* 1,000 *c.* 2,000

3. The sixth-grade class deposited \$29.48 in the school bank in September, \$48.36 in October, and \$38.49 in November. How much did the class deposit during the three months?

a. \$115 *b.* \$85 *c.* \$220

4. Mr. Ford grew 8,168 bushels of corn and Mr. Curtis grew 5,976 bushels of corn. How much more corn did Mr. Curtis grow than Mr. Ford?

a. 3,000 bu. *b.* 2,500 bu. *c.* 2,000 bu.

5. Helen bought 5 pounds of butter at 49¢ a pound, 2 loaves of bread at 12¢ a loaf, and 10 pounds of sugar at $7\frac{1}{2}$¢ a pound. How much did she spend for all of these groceries?

a. \$1.75 *b.* \$3.50 *c.* \$6.15

11. Silent Reading Exercise in Arithmetic to Develop Comprehension.

(Prepared by Agnes Raddatz, Fulton School, Minneapolis.)

Jimmy Raymond's Finances

It was Saturday, July tenth, and Jimmy's fourteenth birthday. Though his mother had given him a party and had invited his five best pals, still Jimmy was bitterly disappointed. For weeks he had been admiring the Ranger Coaster bicycle in Warner's window and had given broad hints to his father—but, the bicycle cost \$48.00 and it was still in the window. Yet, even as Jimmy kicked his heels against the steps a plan was taking shape in his young mind. He received an allowance of \$2.00 weekly. If he could only get the position advertised at the drug store for an errand boy, he might in a few weeks save enough to buy the Ranger himself. With an air of a very busy business man, Jimmy betook himself to the drug store, and, to make a long story short, secured the place. He agreed to work from four to six on school days, and from nine to five on Saturdays, with an hour at noon for lunch, for the princely sum of

$4.00 per week. Already Jimmy had visions of coasting up hill and down dale.

With his heart set on securing the Ranger, Jimmy became very saving, but having a very healthy appetite he couldn't resist spending a quarter for candy, and again as much for a movie, on the Sunday following his birthday. This, of course, diminished his allowance for that week and as he hadn't saved anything before he felt he had spent quite enough; so he surprised his family by staying home that evening.

Monday after school, as Jimmy was hurrying away, Tom shouted to him, "Oh Jimmy, wait for me," but was amazed to hear the answer, "Can't, I'm going to work."

1. If Jimmy saved his full allowance how many weeks would it take him to buy the bicycle?

2. If, instead, he saved all the money he earned, how many weeks would it take?

3. If he saved both his allowance and his salary, how long would it take?

4. For how many hours a week did Jimmy agree to work?

5. If he continued to spend the same amount each Sunday for a month, how much had he spent? (Month of four weeks.)

6. Then how much had he left from his total income for that month?

7. How mucn did Jimmy receive per hour for his work?

8. At that rate how much were his services worth on Saturday alone?

9. If the druggist gave him double pay for work on Sunday, how much would he earn in five hours on Sunday?

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2. Ruch, G. M., *The Objective or New Type Examination* (Chicago: Scott Foresman and Company, 1929).

3. Russell, C., *Classroom Tests* (Boston: Ginn and Company, 1926).

APPENDIX

Arithmetic Tests

1. Brueckner, L. J.—Curriculum Tests in Arithmetic Processes. Grades III, IV, V, VI, VII, VIII. The John C. Winston Company, Philadelphia, Pennsylvania.
2. Brueckner, L. J.—Diagnostic Test in Whole Numbers. Grades 2 to 6. Educational Test Bureau, Minneapolis, Minn.
3. Brueckner, L. J.—Diagnostic Test in Fractions. Grades 5 to 8, all processes. Educational Test Bureau, Minneapolis, Minn.
4. Brueckner, L. J.—Diagnostic Test in Decimals. Grades 5 to 8, all processes. Educational Test Bureau, Minneapolis, Minn.
5. Buckingham Scale for Problems in Arithmetic. Public School Publishing Co., Bloomington, Ill.
6. Buswell and John—Diagnostic Chart for Fundamentals of Arithmetic. Public School Publishing Company, Bloomington, Ill.
7. Cleveland Survey Tests in Arithmetic. Grades II to VIII. Public School Publishing Company, Bloomington, Ill.
8. Compass Survey Tests in Arithmetic. Grades II to VIII. Scott, Foresman and Co., Chicago, Ill. Also distributed by Bureau of Educational Research and Service, University of Iowa, Iowa City, Ia.
9. Courtis Standard Research Tests, Series B. Grades IV to VIII. S. A. Courtis, 1807 E. Grand Blvd., Detroit, Mich.
10. Knight-McClure Arithmetic Neatness Scale. Scott, Foresman and Co., Chicago, Ill.
11. Los Angeles Diagnostic Arithmetic Tests. Grades II to VIII. Research Service Co., 4259 South Van Buren Place, Los Angeles, Cal.

12. Lunceford Diagnostic Tests in Addition. Primary Grades. Bureau of Educational Measurements and Standards, Kansas State Normal School, Emporia, Kans.
13. Monroe Diagnostic Tests in Arithmetic. Grades IV to VIII. Public School Publishing Co., Bloomington, Ill.
14. Monroe General Survey Scales in Arithmetic. Grades III to VIII. Public School Publishing Co., Bloomington, Ill.
15. Monroe Standardized Reasoning Tests in Arithmetic. Grades IV and VIII. Public School Publishing Co., Bloomington, Ill.
16. Otis Reasoning Test in Arithmetic. Grades IV to IX. World Book Company, Yonkers-on-Hudson, New York, and Chicago, Ill.
17. Peet-Dearborn Progress Tests in Arithmetic. Intermediate Series, Grades IV and V; Upper-Grade Series, Grades VI, VII, and VIII. Houghton Mifflin Company, Chicago, Ill.
18. Reavis and Breslich Diagnostic Tests in the Fundamental Operations of Arithmetic and in Problem Solving. Grades VII to IX. University of Chicago Press, Chicago, Ill.
19. Spencer Diagnostic Test in Arithmetic. Grades III to VIII. Bureau of Educational Research, University of Oregon, Eugene, Ore.
20. Stanford Achievement Test. World Book Company, Yonkers-on-Hudson, New York, and Chicago, Ill.
21. Stevenson Problem Analysis Test. Grades IV to IX. Public School Publishing Co., Bloomington, Ill.
22. Stone Reasoning Test (Revised). Grades V to VIII. Bureau of Publications, Teachers College, Columbia University, New York City.
23. Woody-McCall Mixed Fundamentals. Grades II to VIII. Bureau of Publications, Teachers College, Columbia University, New York City.
24. Wisconsin Inventory Tests. Houghton Mifflin Company, Chicago, Ill.

Arithmetic Drill and Remedial Materials

1. Arithmetic Work Books, Ruch, G. M., Knight, F. B., Studebaker, J. W. Grades III to VIII. Scott, Foresman and Co., Chicago, Ill.
2. Brueckner, L. J., Anderson, C. J., Banting, G. O., Merton, Elda. Diagnostic Tests and Remedial Exercises. A series of exercises, one for each grade from 3 to 8, containing survey tests, diagnostic tests, and remedial exercises in all processes. The John C. Winston Company, Philadelphia.
3. Courtis Standard Practice Exercises in Arithmetic. A series of 48 lessons on cards, covering the operations with whole numbers in grades 4 to 8. World Book Company, Yonkers-on-Hudson, N. Y.
4. Economy Remedial Exercises in Whole Numbers (Remedial). Forty-one specially designed exercises on cards permitting continuous and repeated use. Standardized for use in grades 4 to 8. Scott, Foresman and Co., Chicago, Ill.
5. Economy Exercises in Problem-Solving in Arithmetic (Remedial). Standardized for use in grades 5 to 8. Scott, Foresman and Co., Chicago, Ill.
6. Fowlkes-Goff Practice Exercises in Arithmetic. Designed for use in grades 7 to 9. The Macmillan Company, New York.
7. Lennes' Work, Drill, and Test Sheets. Laidlaw Brothers, Chicago, Ill.
8. Studebaker's Practice Exercises in Arithmetic. Scott, Foresman and Company, Chicago, Ill.
9. Osburn's Corrective Exercises in the Fundamentals of Arithmetic (Remedial). Houghton Mifflin Company, Boston, Mass.
10. Thorndike Practice Books in Arithmetic. Rand, McNally and Co., Chicago, Ill.

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